ELEMENTARY CALCULUS

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ELEMENTARY CALCULUS

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PREFACE

The object of this book is to present the elements of the Differential Calculus in a form easily accessible for the undergraduate. It is possible, from the very beginning, to illustrate the ideas and methods of the Calculus by means of applications to physics and geometry, which the student can readily grasp, and which will seem to him of interest and value. To do this, the stress in the illustrative examples worked in the text must be laid first of all on the thought which underlies the method of solution, in distinction from the exposition of a process, reduced in the worst teaching to rules, whereby the answer can be obtained. The treatment of maxima and minima, Chapter III, §§ 2, 3, and curve tracing, Chapter III, § 5 and Chapter VII, § 10, will serve to show what is here meant.

It is, however, also essential that the student receive thorough training in the formal processes and the technique of the Calculus, and this side has been treated with care and completeness. Note, for example, the differentiation of composite functions in Chapter II, § 8, and the exposition of the use of differentials in differentiating in Chapter IV, §§ 4, 5.

An important application of the graphical methods, with which the Calculus is so intimately associated, is that of solving approximately numerical equations which do not come under the standard rules of algebra and trigonometry. Hitherto, however, little attempt has been made to present this subject, simple as it is, in any systematic and elementary manner. In Chapter VII the common methods in use by physicists and others who apply the Calculus are set forth and illustrated by simple examples.

The book might have included a brief treatment of curvature and evolutes, and the cycloid. But probably most

teachers of the Calculus will prefer to take up integration next, and so the closing chapter is devoted to the last of the elementary functions, the inverse trigonometric functions, with special reference to their one great application in the elements of mathematics, namely, their application to integration.

The book is so written that it can be adapted, if desired, to an abridged course, in which, after the fundamentals of the first three chapters have been covered, any of the remaining topics can be treated briefly, and thus a wide scope in subject matter is possible, even when the time is short.

CAMBRIDGE, MASSACHUSETTS,
January, 1921.

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CALCULUS

CHAPTER I

INTRODUCTION

THE Calculus was invented in the seventeenth century by the mathematician, astronomer, and physicist, Sir Isaac Newton in England, and the philosopher Leibniz in Germany. The reaction of the invention on geometry and mathematical physics was most important. In fact, by far the greatest part of the mathematics and the physics of the present day owes its existence to this invention.

1. Functions. The word *function*, in mathematics, was first applied to an expression involving one or more letters which represent variable quantities; as, for example, the expressions

(*a*) $$x^3, \qquad 2x^3 - 3x + 1;$$

(*b*) $$\sqrt{x}, \qquad \sqrt{a^2 - x^2};$$

(*c*) $$\frac{x^2}{a+x}, \qquad \frac{xy}{x^2+y^2}, \qquad \frac{ax+by}{\sqrt{x^2+y^2+z^2}};$$

(*d*) $$\sin x, \qquad \log x, \qquad \tan^{-1} x.$$

In the second example under (*b*), two letters enter; but a is thought of as chosen in advance and then held fast, x alone being variable. A quantity of this kind is called a *constant*. Thus

$$ax + b$$

is a function of x which depends on two constants, a and b.

Such expressions are written in symbolic, or abbreviated, form as $f(x)$, $f(x, y)$ (read: "f of x," "f of x and y" etc.); other letters in common use being F, ϕ, Φ, etc.* Thus the equation

$$f(x) = 2x^3 - 3x + 1 \tag{1}$$

defines the function $f(x)$ in the present case to be $2x^3-3x+1$. Again,

$$\phi(x, y, z) = x^2 + y^2 + z^2 \tag{2}$$

is an equation defining the function $\phi(x, y, z)$ as $x^2 + y^2 + z^2$.

We shall be concerned for the present with functions of one single variable, as illustrated by (1) above. Here, x is called the *independent* variable, since we assign to it any value we like. The value of the function, or more briefly, the *function*, is called the *dependent variable*, and is often denoted by a single letter, as

$$y = f(x)$$

or

$$y = 2x^3 - 3x + 1.$$

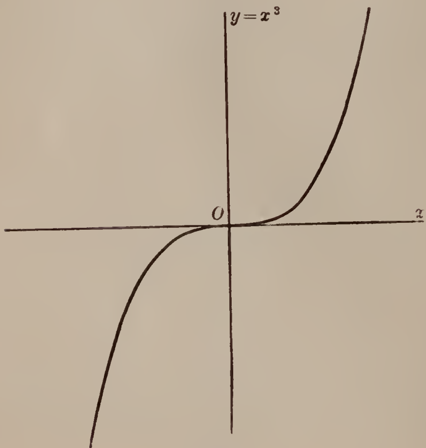

FIG. 1

Graphs. A function of a single variable,

$$y = f(x),$$

can be represented geometrically by its graph, and this representation is of great aid in studying the properties of the function. The independent variable is laid off as the x-coordinate, or abscissa, and the dependent variable, or func-

* To distinguish between $f(x)$ and $F(x)$, read the first "small f of x" and the second, "large F of x."

tion, as the y-coordinate, or ordinate. Thus the graph of the function

$$f(x) = x^3$$

is the curve

$$y = x^3.$$

Illustrations from Geometry and Physics. The familiar formulas of geometry and physics afford simple examples of functions. Thus the area, A, of a circle is given by the formula

$$A = \pi r^2,$$

where r denotes the radius, π being the fixed number 3.1416. Here, r is thought of as the independent variable, — it may have any positive value whatever, — and A is the function, or dependent variable.

Again, for the three round bodies, the volumes are:

(a) $$V = \tfrac{4}{3}\pi r^3,$$ sphere;

(b) $$V = \pi r^2 h,$$ cylinder;

(c) $$V = \frac{\pi}{3} r^2 h,$$ cone.

In (b) and (c), h denotes the altitude and r, the radius of the base; V is here a function of the *two* independent variables, r and h.

The surfaces of these bodies are given by the formulas:

(α) $$S = 4\pi r^2,$$ sphere;

(β) $$S = 2\pi rh,$$ cylinder;

(γ) $$S = \pi rl,$$ cone;

l, in the last formula, denoting the slant height. Thus we have three further examples of functions of one or of two variables.

The formula for a freely falling body is

$$s = \tfrac{1}{2} g t^2,$$

where s denotes the distance fallen and t the time; g is a constant, for it has just one value after the units of time and

length have been chosen. Here, t is the independent variable and s is the function. If, however, we solve this equation for t:

$$t = \sqrt{\frac{2s}{g}},$$

then s becomes the independent variable and t, the function.

Sometimes two variables are connected by an equation, as

$$pv = c,$$

where p denotes the pressure of a gas and v its volume, the temperature remaining constant. Here, either variable can be chosen as the independent variable, and when the equation is solved for the other variable, the latter becomes the dependent variable, or function. Thus, if we write

$$v = \frac{c}{p},$$

p is the independent variable, and v is expressed as a function of p. But if we write

$$p = \frac{c}{v},$$

the rôles are reversed.

The Independent Variable Restricted. Often the independent variable is restricted to a certain interval, as in the case of the function

$$y = \sqrt{a^2 - x^2}.$$

Here, x must lie between $-a$ and a:

$$-a \leqq x \leqq a,$$

since other values of x make $a^2 - x^2$ negative, and the above expression has no meaning.

This was also the case with the geometric examples above cited. There, r, h, l were necessarily positive, since there is no such thing, for example, as a sphere of zero or negative radius.

The independent variable may also be restricted to being a positive whole number, as in the case of the sum of the first n

terms of a geometric progression:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Here,

$$s_n = \frac{a - ar^n}{1 - r}.$$

Suppose $a = 1$, $r = \frac{1}{2}$, the progression thus becoming

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}.$$

Then

$$s_n = \frac{1 - \dfrac{1}{2^n}}{1 - \dfrac{1}{2}} = 2 - \frac{1}{2^{n-1}},$$

and we have an example of a function with the independent variable a natural number, *i.e.* a positive integer.

In the case of the functions treated in the calculus, the domain of the independent variable is a *continuum*, *i.e.*, for functions of a single variable, an interval, as

$$a \leqq x \leqq b, \qquad \text{or} \qquad 0 < x.$$

Ordinarily, the later letters of the alphabet, particularly x, y, z, are used to represent variables, the early letters denoting constants. Thus it will be understood, when such an expression as

$$ax^2 + bx + c$$

is written down, that a, b, c are constants and x is the variable.

Multiple-Valued Functions; Principal Value. The expressions above cited are all examples of *single-valued* functions; *i.e.* to each value of the independent variable x corresponds but one value of the function. A function may, however, be *multiple-valued;* as in the case of the function y defined by the equation

$$x^2 + y^2 = a^2.$$

Here

$$y = \pm\sqrt{a^2 - x^2},$$

and so is a double-valued function. This function is, however, completely represented by means of the two single-valued functions,

$$y=\sqrt{a^2-x^2} \qquad \text{and} \qquad y=-\sqrt{a^2-x^2}.$$

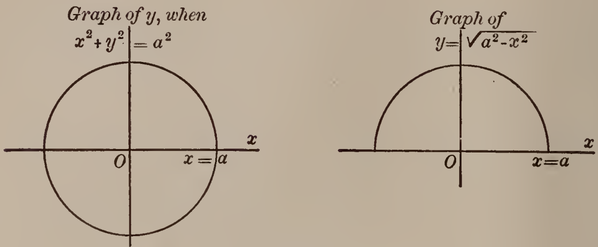

FIG. 2

They form the *branches* of this multiple-valued function.

The student should notice that the radical sign $\sqrt{\ }$ is defined as meaning the *positive* square root, NOT *either* the positive *or* the negative square root at pleasure. If it is desired to express the negative square root, the minus sign must be written in front of the radical sign, $-\sqrt{\ }$. Thus $\sqrt{4}=2$, and not -2. This does not mean that 4 has only one square root. It means that the notation $\sqrt{4}$ calls for the positive, and not for the negative, of these two roots.

Again,

$$\sqrt{(-2)^2}=2,$$

and not -2. For $(-2)^2=4$, and $\sqrt{\ }$ means the positive root. And, generally,

$$(1) \qquad \begin{cases} \sqrt{x^2}=x, & \text{if } x \text{ is positive;} \\ \sqrt{x^2}=-x, & \text{if } x \text{ is negative.} \end{cases}$$

A similar remark applies to the symbol $\sqrt[2n]{\ }$, which is likewise used to mean the positive $2n$th root. Moreover,

$$a^{\frac{1}{2}}=\sqrt{a}, \qquad a^{\frac{1}{2n}}=\sqrt[2n]{a}.$$

The function

$$y=\sqrt{x}$$

is often called the *principal value* of the double-valued function defined by the equation

$$y^2 = x.$$

Since multiple-valued functions are studied by means of single-valued functions, it will be understood henceforth, unless the contrary is explicitly stated, that the word *function* means *single-valued* function.

Absolute Value. It is frequently desirable to use merely the *numerical*, or *absolute value* of a quantity, and to have a notation for the same. The notation is: $|x|$, read "absolute value of x." Thus

$$|-3| = 3 \qquad \text{and} \qquad |3| = 3.$$

We can now write in a single formula what was formerly stated by the two equations (1), namely the definition of the radical sign, $\sqrt{\ }$:

$$\sqrt{a^2} = |a|. \tag{2}$$

Again, by the *difference* of two numbers we often mean the value of the larger less the smaller. Thus the difference of 4 and 10 is 6; and the difference of 10 and 4 is also 6. The difference of a and b, in this sense, can be expressed as either

$$|b - a| \qquad \text{or} \qquad |a - b|.$$

Continuous Functions. A function, $f(x)$, is said to be *continuous* if a slight change in x produces but a slight change in the value of the function. Thus the polynomials are readily shown to be continuous; cf. Chap. II, § 5, and all the functions with which we shall have to deal are continuous, save at exceptional points.

As an example of a function which is discontinuous at a certain point may be cited the function (see Fig. 3)

$$f(x) = \frac{1}{x}.$$

When x approaches the value 0, the function increases numerically without limit. The graph of the function has the axis of y as an asymptote.

The fractional rational functions are continuous except at the points at which the denominator vanishes.

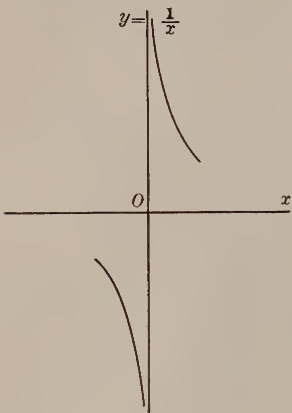

FIG. 3

Thus the function

$$f(x) = \frac{x^2+1}{x^2-1}$$

is continuous except at the points $x = 1$ and $x = -1$. Here, the function becomes infinite. Its graph is the curve

$$y = \frac{x^2+1}{(x-1)(x+1)},$$

which evidently has the lines $x = 1$ and $x = -1$ as asymptotes.

The function

$$f(x) = \tan x$$

is continuous except when x is an odd multiple of $\pi/2$,

$$x = \frac{2n+1}{2}\pi.$$

EXERCISES

1. If
$$f(x) = x^2 - 4x + 3,$$
show that $f(1) = 0, \qquad f(2) = -1, \qquad f(3) = 0.$

Compute $f(0)$, $f(4)$. Plot the graph of the function.

2. If
$$\phi(x) = 4x^3$$
compute $\phi(2)$ and $\phi(\sqrt{3})$.

3. If
$$F(x) = \frac{2x-3}{x+7},$$
compute $F(\sqrt{2})$ correct to three significant figures.

Ans. $-.0204$.

4. If
$$\Phi(x) = (x^3 - x)\sin x,$$
find all the values of x for which
$$\Phi(0) = 0.$$

5. If
$$\psi(x) = x^{\frac{2}{3}} - x^{-\frac{2}{3}},$$
find $\psi(8)$.

6. Solve the equation
$$x^3 - xy + 3 = 5y$$
for y, thus expressing y as a function of x.

7. If
$$f(x) = a^x,$$
show that
$$f(x)f(y) = f(x+y).$$

8. If
$$y = \frac{x+1}{2x-3},$$
express x as a function of y.

9. Draw the graph of the function
$$f(x) = x^2 + 4x + 3,$$
taking 1 cm. as the unit.

Suggestion: Write the function in the form, $(x+1)(x+3)$.

10. Draw the graph of the function
$$f(x) = x^3 - 4x.$$

11. Draw the graph of the function
$$f(x) = \frac{x}{x^2 - 4},$$
and hence illustrate the two discontinuities which this function has.

12. Draw the graph of the function
$$f(x) = \frac{1}{x^2} - \frac{1}{(x-1)^3}.$$

13. For what values of x are the following functions discontinuous?

(*a*) $f(x) = \cot x$; (*c*) $f(x) = \csc x$;

(*b*) $f(x) = \sec x$; (*d*) $f(x) = \tan \frac{x}{2}$.

14. Express the double-valued function defined by the equation
$$x^2 - y^2 = -1$$
in terms of two single-valued functions.

15. Express the quadruple-valued function defined by the equation
$$y^4 - 2y^2 + x^2 = 0$$
in terms of four single-valued functions.

16. Express the sum s_n of the first n terms of the arithmetic progression
$$a + (a + b) + (a + 2b) + \cdots + (a + \overline{n-1}b)$$
as a function of n.

Thus obtain the sum of the first n positive integers as a function of n.

17. If P dollars are put at simple interest for one year at r per cent, (*a*) express the amount A (principal and interest) as a function of P and r. (*b*) Express the amount A at the end of n years, the interest being compounded annually, as a function of P, r, and n. (*c*) Express the amount A at the end of one year, if the interest is compounded m times in the year at equal intervals, as a function of P, r, m.

2. Continuation. General Definition of a Function. The conception of the function is broader than that of the mathematical formulas mentioned in the last paragraph. Let us state the definition in its most general form.

Definition of a Function. *The variable y is said to be a function of the variable x if there exists a law whereby, when x is given, y is determined.*

Consider, for example, a quantity of gas confined in a chamber, — for instance, the charge of the mixture of gasolene and air as it is being compressed in the cylinder of an automobile. The charge exerts at each instant a definite pressure, p, of so many pounds per square inch on the walls of the chamber, and this pressure varies with the volume, v, occupied by the charge. In the small fraction of a second under consideration, presumably but little heat is gained or lost through the walls of the chamber, and thus p is a function of v,

$$p = f(v).$$

In this case, the function is given approximately by the mathematical formula

$$p = \frac{C}{v^{1.4}},$$

where C denotes a certain constant. But that which is of first importance for our conception is not the formula, but the fact that *to each value of v there corresponds a definite value of p.* In other words, there is a definite graph of the relation between v and p. The representation of the relation by a mathematical formula is, indeed, important; but what we must first see clearly is the fact that there is a definite relation to express.

As another illustration take the curve traced out by the pen of a self-registering thermometer of the kind used at a meteorological station. The instrument consists of a cylindrical drum turned slowly by clockwork at uniform speed about a vertical axis, a sheet of paper being wound firmly round the drum. A pen is held against the paper, and the height of the pen above a certain level is proportional to the height of the temperature above the temperature corre-

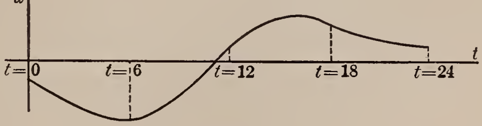

FIG. 4

sponding to that level. The apparatus is set in operation, and when the drum has been turning for a day, the paper is taken off and spread out flat. Thus we have before us the graph of the temperature for the day in question, the independent variable being the time (measured in hours from midnight) and the dependent variable being the temperature, represented by the other coordinate of a point on the curve.

One more illustration, — that of the resistance of the atmosphere to a rifle bullet. This resistance, measured in pounds, depends on the velocity of the bullet, and it is a matter of physical experiment to determine the law. But that which is of first importance for our conceptions is the fact *that there is a law*, whereby, when the velocity, v, is given an arbitrary value within the limits of the velocities considered, there corresponds to this v a definite value, R, of the resistance. We say, then, that R is a *function* of v and write

$$R = f(v).$$

In this connection, cf. the chapter on Mechanics, § 7, Graph of the Resistance, in the author's *Differential and Integral Calculus*.

CHAPTER II

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS
GENERAL THEOREMS

1. Definition of the Derivative. The Calculus deals with varying quantity. If y is a function of x, then x is thought of, not as having one or another special value, but as flowing or growing, just as we think of time or of the expanding circular ripples made by a stone dropped into a placid pond. And y varies with x, sometimes increasing, sometimes decreasing. Now if we consider the change in x for a short interval, say from $x = x_0$ to $x = x'$, the corresponding change in y, as y goes from y_0 to y', will be in general almost proportional to the change in x. For the *ratio* of these changes is

$$\frac{y' - y_0}{x' - x_0},$$

and this quantity changes only slightly when x' is nearly equal to x_0. Let us study this last statement minutely.

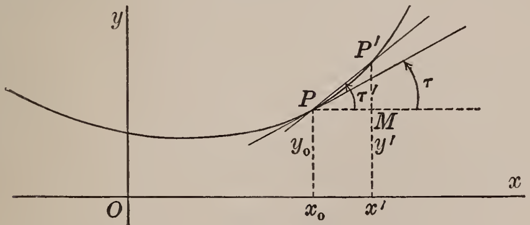

FIG. 5

The above ratio has a simple geometric meaning, if we draw the graph of the function; for

$$PM = x' - x_0; \qquad MP' = y' - y_0,$$

and
$$\frac{y'-y_0}{x'-x_0}=\tan\tau',$$

where τ' denotes the angle which the secant PP' makes with the axis of x. Now let x' approach x_0 as its limit. Then τ' approaches as its limit the angle τ which the tangent line of the graph at P makes with the axis of x, and hence

$$\lim_{x' \doteq x_0} \frac{y'-y_0}{x'-x_0}=\tan\tau$$

$\left(\text{read: "limit, as } x' \text{ approaches } x_0 \text{, of } \dfrac{y'-y_0}{x'-x_0}\text{"}\right)$.

The determination of this limit and the discussion of its meaning is the fundamental problem of the Differential Calculus.

Such are the concepts which underlie the idea of the derivative of a function. We turn now to a precise formulation of the definition. Let

(1) $$y=f(x)$$

be a given function of x. Let x_0 be an arbitrary value of x, and let y_0 be the corresponding value of the function:

(2) $$y_0=f(x_0).$$

Give to x an increment,* Δx; *i.e.* let x have a new value, x', and denote the change in x, namely, $x'-x_0$, by Δx:

$$x'-x_0=\Delta x, \qquad x'=x_0+\Delta x.$$

The function, y, will thereby have changed to the value

(3) $$y'=f(x')$$

and hence have received an increment, Δy, where

$$y'-y_0=\Delta y, \qquad y'=y_0+\Delta y.$$

* The student must not think of this symbol as meaning Δ *times* x. We might have used a single letter, as h, to represent the difference in question: $x'=x_0+h$; but h would not have reminded us that it is the increment of x, and not of y, with which we are concerned. The notation is read "*delta x*."

Equation (3) is equivalent to the following:

(4) $$y_0 + \Delta y = f(x_0 + \Delta x).$$

From equations (2) and (4) we obtain by subtraction the equation

$$\Delta y = f(x_0 + \Delta x) - f(x_0),$$

and hence

(5) $$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

DEFINITION OF A DERIVATIVE. *The limit which the ratio* (5), *namely* $\frac{\Delta y}{\Delta x}$, *approaches when* Δx *approaches zero:*

(6) $$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} \qquad \text{or} \qquad \lim_{\Delta x \doteq 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x},$$

is called the derivative of y with respect to x and is denoted by $D_x y$ *or* $D_x f(x)$ (read: "*D x* of *y*"):

(7) $$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = D_x y.$$

In this definition Δx may be negative as well as positive, and the limit (6) must be the same when Δx approaches 0 from the negative side as when it approaches 0 from the positive side.

To *differentiate* a function is to find its derivative.

The geometrical interpretation of the analytical process of differentiation is to find the slope of the graph of the function. For,

$$\tan \tau' = \frac{\Delta y}{\Delta x}$$

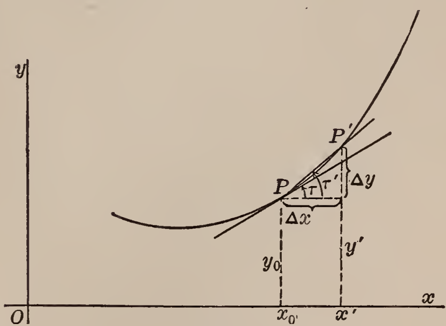

FIG. 6

and

$$\tan \tau = \lim_{P' \doteq P} \tan \tau' = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = D_x y.$$

2. Differentiation of x^n. Suppose n has the value 3, so that it is required to differentiate the function

(1) $$y = x^3.$$

We must follow the definition of § 1 step by step. Begin, then, by assigning to x a particular value, x_0, which is to be held fast during the rest of the process, and compute from equation (1) the corresponding value y_0 of y:

(2) $$y_0 = x_0^3.$$

Next, give to x an arbitrary increment, Δx, denote the corresponding increment in y by Δy, and compute it. To this end we first write down the equation

(3) $$y_0 + \Delta y = (x_0 + \Delta x)^3.$$

The right-hand side of this equation can be expanded by the binomial theorem, and hence (3) can be written in a new form :*

(4) $$y_0 + \Delta y = x_0^3 + 3\, x_0^2 \Delta x + 3\, x_0 \Delta x^2 + \Delta x^3.$$

Subtract equation (2) from equation (4):

$$\Delta y = 3\, x_0^2 \Delta x + 3\, x_0 \Delta x^2 + \Delta x^3.$$

Next, divide through by Δx:

$$\frac{\Delta y}{\Delta x} = 3\, x_0^2 + 3\, x_0 \Delta x + \Delta x^2.$$

We are now ready to let Δx approach 0 as its limit:

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} (3\, x_0^2 + 3\, x_0 \Delta x + \Delta x^2).$$

* It is at this point that the specific properties of the function x^3 come into play. Here, it is the binomial theorem that enables us ultimately to compute the limit. In the differentiations of later paragraphs and chapters it will always be some characteristic property of the function in hand which will make possible a transformation at this point.

The limit of the left-hand side is, by definition, $D_x y$. On the right-hand side, each of the last two terms in the parenthesis approaches the limit 0, and so their sum approaches 0, also.

The first term does not change with Δx. Hence, the whole parenthesis approaches the limit $3x_0^2$. We have, then, as the final result:

$$D_x y = 3x_0^2.$$

The subscript has now served its purpose, which was, to remind us that x_0 is not to vary with Δx, and it may be dropped. Thus

$$D_x x^3 = 3x^2.$$

The differentiation of the function x^n in the general case that n is any positive integer can be carried through in precisely the same manner. As the result of the first step we have

$$y_0 = x_0^n. \tag{5}$$

Next comes:

$$y_0 + \Delta y = (x_0 + \Delta x)^n, \tag{6}$$

and we now apply the binomial theorem to the expression on the right-hand side. Thus

$$y_0 + \Delta y = x_0^n + nx_0^{n-1}\Delta x + \frac{n(n-1)}{1 \cdot 2}x_0^{n-2}\Delta x^2 + \cdots + \Delta x^n. \tag{7}$$

On subtracting (5) from (7) we have:

$$\Delta y = nx_0^{n-1}\Delta x + \frac{n(n-1)}{1 \cdot 2}x_0^{n-2}\Delta x^2 + \cdots + \Delta x^n.$$

Now divide through by Δx:

$$\frac{\Delta y}{\Delta x} = nx_0^{n-1} + \frac{n(n-1)}{1 \cdot 2}x_0^{n-2}\Delta x + \cdots + \Delta x^{n-1},$$

and let Δx approach the limit zero:

$$\lim_{\Delta x \doteq 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0}\left(nx_0^{n-1} + \frac{n(n-1)}{1 \cdot 2}x_0^{n-2}\Delta x + \cdots + \Delta x^{n-1}\right).$$

Each term of the parenthesis after the first is the product of a constant factor and a positive power of Δx. This second

factor approaches zero when Δx approaches zero; consequently the whole term approaches zero. There is only a fixed number of these terms, and so the whole parenthesis approaches the limit nx_0^{n-1}. Hence

$$D_x y = nx_0^{n-1}.$$

On dropping the subscript we obtain the final result:

(8) $$D_x x^n = nx^{n-1}.$$

In particular, if $n = 1$, we have

(9) $$D_x x = 1.$$

EXERCISES

Differentiate the following seven functions, applying the process of § 1 step by step.

1. $y = 4x^3$. *Ans.* $D_x y = 12x^2$.

2. $y = x^4$.

3. $y = 2x^2 - 3x + 1$. *Ans.* $D_x y = 4x - 3$.

4. $y = x^7 - x^5$. *Ans.* $D_x y = 7x^6 - 5x^4$.

5.* $f(x) = 1 - 2x^4$. *Ans.* $D_x f(x) = -8x^3$.

6. $\phi(x) = x^2 - 2x + 1$.

7. $F(x) = (1 - x)^2$.

8. Let $$y = 5x - x^2,$$

and take $x_0 = 1$; then $y_0 = 4$. If $\Delta x = .2$, then $\Delta y = .56$ and $\frac{\Delta y}{\Delta x} = 2.8$. Show further that,

for $\Delta x = .1$, $\Delta y = .29$, $\frac{\Delta y}{\Delta x} = 2.9$;

and

for $\Delta x = .01$, $\Delta y = .0299$, $\frac{\Delta y}{\Delta x} = 2.99$.

* It is immaterial whether we write

$$y = 1 - 2x^4 \qquad \text{or} \qquad f(x) = 1 - 2x^4.$$

Plot the curve accurately for values of x from $x = 0$ to $x = 5$, taking 1 cm. as the unit, and draw the secants * in each of the three foregoing cases.

What appears to be the slope of the curve at the point $(x_0, y_0) = (1, 4)$? Prove your guess to be correct.

9. In Ex. 7, let $x_0 = -1$. If Δx is given successively the values .01 and $-.01$, compute Δy and $\dfrac{\Delta y}{\Delta x}$.

10. Complete the following table:

Δx	Δy	$\tan \tau' = \dfrac{\Delta y}{\Delta x}$
.1		
.01		
.001		

for each of the functions:

(a) $y = x^2 - 2x + 1,$ $x_0 = 2;$

(b) $y = x - x^3,$ $x_0 = -1;$

(c) $y = 3x^2 - x,$ $x_0 = 0.$

11. By means of the general theorem (8) write down the derivatives of the following functions:

$$x^4; \quad x^5; \quad x^{10}; \quad x; \quad x^{99}.$$

By means of the definition of § 1 differentiate each of the following functions:

12. $y = \dfrac{1}{x}.$ *Ans.* $D_x y = -\dfrac{1}{x^2}.$

13. $y = \dfrac{1}{x^2}.$ *Ans.* $D_x y = -\dfrac{2}{x^3}.$

14. $y = \dfrac{1}{x^3}.$ *Ans.* $D_x y = -\dfrac{3}{x^4}.$

* The student should recall from his earlier work how to draw a straight line on squared paper when a point and the slope of the line are given.

3. Derivative of a Constant. The function

$$f(x) = c,$$

where c denotes a constant, has for its graph a right line parallel to the axis of x. Since the derivative of a function is represented geometrically by the slope of its graph, it is clear that the derivative of this function is zero:

$$D_x c = 0.$$

It is instructive, however, to obtain this result analytically by the process of § 1. We have here:

$$y_0 = f(x_0) = c,$$

$$y_0 + \Delta y = f(x_0 + \Delta x) = c;$$

hence $$\Delta y = 0 \qquad \text{and} \qquad \frac{\Delta y}{\Delta x} = 0.$$

Now allow Δx to approach 0. The value of $\Delta y/\Delta x$ is always 0, and hence its limit* is 0:

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = 0, \qquad \text{or} \qquad D_x c = 0.$$

* We note here an error frequently made in presenting the subject of limits in school mathematics. It is there often stated that "a variable X approaches a limit A if X comes indefinitely near to A, but *never reaches* A." This last requirement is not a part of the conception of a variable's approaching a limit. It is true that it is often inexpedient to allow the *independent variable* to reach its limit. Thus, in differentiating a function, the ratio $\Delta y/\Delta x$ ceases to have a meaning when $\Delta x = 0$, since division by 0 is impossible. The problem of differentiation is not to find the value of $\Delta y/\Delta x$ when $\Delta x = 0$; such a question would be absurd. What we do is to allow Δx to approach zero as its limit without ever reaching that limit. We can do this for the reason that Δx is the *independent variable*.

When, however, it is Δy or $\Delta y/\Delta x$ that is under consideration, we have to do with *dependent variables*, and we have no control over them, as to whether they reach their limit or not. Thus in the case of the text both Δy and $\Delta y/\Delta x$ are constants ($= 0$). When Δx approaches 0, they always have one and the same value, and so, under the correct conception of approach to a limit each approaches a limit, namely 0.

We can state the result by saying: *The derivative of a constant is* 0.

4. Differentiation of $\sqrt{x}$. Let us differentiate

$$y = \sqrt{x}.$$

Here, $\qquad y_0 = \sqrt{x_0}, \qquad y_0 + \Delta y = \sqrt{x_0 + \Delta x},$

$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x_0 + \Delta x} - \sqrt{x_0}}{\Delta x}.$$

We cannot as yet see what limit the right-hand side approaches when Δx approaches 0, for both numerator and denominator approach 0, and $\frac{0}{0}$ has no meaning. We can, however, transform the fraction by multiplying numerator and denominator by the *sum* of the radicals and recalling the formula of Elementary Algebra:

$$a^2 - b^2 = (a - b)(a + b).$$

Thus $$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x_0 + \Delta x} - \sqrt{x_0}}{\Delta x} \cdot \frac{\sqrt{x_0 + \Delta x} + \sqrt{x_0}}{\sqrt{x_0 + \Delta x} + \sqrt{x_0}}$$

$$= \frac{1}{\Delta x} \cdot \frac{(x_0 + \Delta x) - x_0}{\sqrt{x_0 + \Delta x} + \sqrt{x_0}} = \frac{1}{\sqrt{x_0 + \Delta x} + \sqrt{x_0}},$$

and hence $$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{1}{\sqrt{x_0 + \Delta x} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}.$$

Dropping the subscript, we have:

$$D_x \sqrt{x} = \frac{1}{2\sqrt{x}}.$$

EXERCISES

1. Differentiate the function $y = \dfrac{1}{\sqrt{x}}$. *Ans.* $D_x y = -\dfrac{1}{2\sqrt{x^3}}$.

2. If $$y = \sqrt{2 - 3x},$$

show that $$D_x y = \frac{-3}{2\sqrt{2 - 3x}}.$$

3. Prove: $$D_x\sqrt{1-x} = -\frac{1}{2\sqrt{1-x}}.$$

4. Prove: $$D_x\sqrt{a+bx} = \frac{b}{2\sqrt{a+bx}}.$$

5. Three Theorems about Limits. Infinity.* In the further treatment of differentiation the following theorems are needed.

THEOREM I. *The limit of the sum of two variables is equal to the sum of their limits:*

$$\lim (X + Y) = \lim X + \lim Y.$$

In this theorem we think of X and Y as two *dependent variables*, each of which approaches a limit:

$$\lim X = A, \qquad \lim Y = B.$$

We do not care what the independent variable may be. In the applications of the theorem to computing derivatives, the independent variable will always be Δx, and it will be allowed to approach 0, without ever reaching its limit.

Since X approaches A, it comes nearer and nearer to this value. Let the difference between the variable and its limit be denoted by ϵ; then the limit of ϵ is 0:

(1) $$X - A = \epsilon, \qquad X = A + \epsilon;$$

$$\lim \epsilon = 0.$$

Similarly, let

(2) $$Y - B = \eta, \qquad Y = B + \eta;$$

then $$\lim \eta = 0.$$

* This paragraph should be read carefully and its content grasped, but the student should not be required to reproduce it at this stage of his work. He will meet frequent applications of its principles, and he should turn back each time to these pages and read anew the theorem involved, with its proof. When he has thus come to see the full meaning and importance of these theorems, he should demand of himself that he be able readily to reproduce the proofs.

It will be convenient to think of these numbers as represented geometrically by points on the scale of numbers, thus:

FIG. 7

Of course, A and B may be negative or 0. ϵ and η may be negative as well as positive, or even 0.

Consider the variable $X+Y$. Its value from (1) and (2) is:

$$X+Y=A+B+\epsilon+\eta.$$

Hence $$\lim(X+Y)=\lim(A+B+\epsilon+\eta).$$

But since $\lim\epsilon=0$ and $\lim\eta=0$, the limit of the right-hand side of this equation is $A+B$, or

$$\lim(X+Y)=A+B.$$

Consequently, $$\lim(X+Y)=\lim X+\lim Y, \qquad \text{q. e. d.}$$

COROLLARY. *The limit of the sum of any fixed number of variables is equal to the sum of the limits of these variables:*

$$\lim(X_1+X_2+\cdots+X_n)=\lim X_1+\lim X_2+\cdots+\lim X_n.$$

Suppose $n=3$. Then

$$X_1+X_2+X_3=(X_1+X_2)+X_3.$$

From Theorem I it follows that

$$\lim(X_1+X_2+X_3)=\lim(X_1+X_2)+\lim X_3.$$

Applying the Theorem again, we have

$$\lim(X_1+X_2)=\lim X_1+\lim X_2.$$

Hence the corollary is true for $n=3$. It can now be established for $n=4$; and so on. By the method of Mathematical Induction it can be proven generally. Or, the proof of the main theorem may be extended directly to the present theorem.

THEOREM II. *The limit of the product of two variables is equal to the product of their limits:*

$$\lim (XY) = (\lim X)(\lim Y).$$

From equations (1) and (2) it follows that

$$XY = (A + \epsilon)(B + \eta),$$

or
$$XY = AB + B\epsilon + A\eta + \epsilon\eta.$$

Hence
$$\lim XY = \lim (AB + B\epsilon + A\eta + \epsilon\eta).$$

Since A and B are constants, each of the last three terms in the parenthesis approaches the limit 0, and so the limit of the parenthesis is AB. Hence

$$\lim (XY) = AB,$$

or
$$\lim (XY) = (\lim X)(\lim Y), \qquad \text{q.e.d.}$$

COROLLARY. *The limit of the product of n variables is equal to the product of the limits of these variables:*

$$\lim (X_1 X_2 \cdots X_n) = (\lim X_1)(\lim X_2) \cdots (\lim X_n).$$

The proof is similar to that of the corollary under Theorem I.

Remark. As a particular case under Theorem II we have:

$$\lim (CX) = C(\lim X),$$

where C is a constant.

THEOREM III. *The limit of the quotient of two variables is equal to the quotient of their limits, provided that the limit of the divisor is not 0:*

$$\lim \frac{X}{Y} = \frac{\lim X}{\lim Y}, \qquad \text{if } \lim Y \neq 0.$$

From equations (1) and (2) above we have:

$$\frac{X}{Y} = \frac{A + \epsilon}{B + \eta}.$$

Subtract A/B from each side of this equation and reduce:

$$\frac{X}{Y}-\frac{A}{B}=\frac{A+\epsilon}{B+\eta}-\frac{A}{B}=\frac{B\epsilon-A\eta}{B^2+B\eta}.$$

Hence
$$\frac{X}{Y}=\frac{A}{B}+\frac{B\epsilon-A\eta}{B^2+B\eta},$$

and
$$\lim\frac{X}{Y}=\lim\left(\frac{A}{B}+\frac{B\epsilon-A\eta}{B^2+B\eta}\right).$$

We wish to show that

$$\lim\frac{B\epsilon-A\eta}{B^2+B\eta}=0.$$

The numerator is seen at once to approach zero. The limit of the denominator is B^2. Let H be a positive number less than

FIG. 8

B^2. Then the denominator will finally become and remain greater than H, and hence the numerical value of the quotient in question will not exceed the numerical value of

$$\frac{B\epsilon-A\eta}{H}.$$

But the limit of this expression is zero, and hence

$$\lim\frac{X}{Y}=\frac{A}{B},$$

or
$$\lim\frac{X}{Y}=\frac{\lim X}{\lim Y}, \qquad \text{q. e. d.}$$

In particular, we see that, *if a variable approaches unity as its limit, its reciprocal also approaches unity:*

If $\quad \lim X=1,\quad$ *then* $\quad \lim\frac{1}{X}=1.$

Also, $$\lim \frac{C}{X} = \frac{C}{\lim X},$$

where C is a constant and $\lim X \neq 0$.

Remark. If the denominator Y approaches 0 as its limit, no general inference about the limit of the fraction can be drawn, as the following examples show. Let Y have the values:

$$Y = \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \cdots, \frac{1}{10^n}, \cdots.$$

(1) If the corresponding values of X are:

$$X = \frac{1}{10^2}, \frac{1}{100^2}, \frac{1}{1000^2}, \cdots, \frac{1}{10^{2n}}, \cdots,$$

then $$\lim \frac{X}{Y} = \lim \frac{1}{10^n} = 0.$$

(2) If $$X = \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{100}}, \frac{1}{\sqrt{1000}}, \cdots, \frac{1}{10^{\frac{n}{2}}}, \cdots,$$

then $X/Y = 10^{n/2}$ approaches no limit, but increases beyond all limit.

(3) If $$X = \frac{c}{10}, \frac{c}{100}, \frac{c}{1000}, \cdots, \frac{c}{10^n}, \cdots,$$

where c is any arbitrarily chosen fixed number, then

$$\lim \frac{X}{Y} = c.$$

(4) If $$X = \frac{1}{10}, -\frac{1}{100}, \frac{1}{1000}, -\frac{1}{10{,}000}, \cdots,$$

then X/Y assumes alternately the values $+1$ and -1, and hence, although remaining finite, approaches no limit.

To sum up, then, we see that when X and Y both approach 0 as their limit, their ratio may approach any limit whatever, or it may increase beyond all limit, or finally, although *remain-*

ing finite, i.e. always lying between two *fixed* numbers, no matter how widely the latter may differ from each other in value, — it may jump about and so fail to approach a limit.

Infinity. If $\lim X = A \neq 0$ and $\lim Y = 0$, then X/Y increases beyond all limit, or *becomes infinite.* A variable Z is said to become infinite when it ultimately becomes and remains greater numerically than any preassigned quantity, however large.* If it takes on only positive values, it *becomes positively infinite;* if only negative values, it *becomes negatively infinite.* We express its behavior by the notation:

$$\lim Z = \infty \quad \text{or} \quad \lim Z = +\infty \quad \text{or} \quad \lim Z = -\infty.$$

But this notation does not imply that infinity is a limit; the variable in this case approaches no limit. And so the notation should not be read "Z approaches infinity" or "Z equals infinity; but "*Z becomes infinite.*"

Thus if the graph of a function has its tangent at a certain point parallel to the axis of ordinates, we shall have for that point:

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \infty;$$

read: "$\Delta y/\Delta x$ becomes infinite when Δx approaches 0."

Some writers find it convenient to use the expression "a variable approaches a limit" to include the case that the variable becomes infinite. We shall not adopt this mode of expression, but shall understand the words "approaches a limit" in their strict sense.

If a function $f(x)$ becomes infinite when x approaches a certain value a, as for example

$$f(x) = \frac{1}{x} \quad \text{for} \quad a = 0,$$

* Note that the statement sometimes made that "Z becomes greater than any assignable quantity" is absurd. There is no quantity that is greater than any assignable quantity.

we denote this by writing

$$f(a)=\infty$$

(or $f(a)=+\infty$ or $=-\infty$, if this happens to be the case and we wish to call attention to the fact).

It is in this sense that the equation

$$\tan 90^\circ=\infty$$

is to be understood in Trigonometry. The equation does *not* mean that 90° *has* a tangent and that the *value* of the latter is ∞. It means that, as x approaches 90° as its limit, $\tan x$ exceeds numerically any number one may name in advance, and stays above this number as x continues to approach 90° without ever reaching its limit, 90°.

Definition of a Continuous Function. We can now make more explicit the definition given in Chapter I by saying: $f(x)$ is continuous at the point $x=a$ if

$$\lim_{x \doteq a} f(x)=f(a).$$

From Exercises 1–3 below it follows that the polynomials are continuous for all values of x, and that the fractional rational functions are continuous except when the denominator vanishes.

EXERCISES

1. Show that, if n is any positive integer,

$$\lim (X^n)=(\lim X)^n.$$

2. If $$G(x)=c_0+c_1x+c_2x^2+\cdots+c_nx^n,$$

then $$\lim_{x \doteq a} G(x)=G(a)=c_0+c_1a+c_2a^2+\cdots+c_na^n.$$

3. If $G(x)$ and $F(x)$ are any two polynomials and if $F(a)\neq 0$,

then $$\lim_{x \doteq a} \frac{G(x)}{F(x)}=\frac{G(a)}{F(a)}.$$

4. If X remains finite and Y approaches 0 as its limit, show that

$$\lim (XY) = 0.$$

5. Show that $$\lim_{x=\infty} \frac{x^2+1}{3x^2+2x-1} = \frac{1}{3}.$$

Suggestion. Begin by dividing the numerator and the denominator by x^2.

Evaluate the following limits:

6. $\lim\limits_{x=\infty} \dfrac{x+1}{x^3-7x+3}$.

7. $\lim\limits_{x=\infty} \dfrac{12x^6+5}{4x^6+3x^4+7x^2-1}$.

8. $\lim\limits_{x=\infty} \dfrac{ax+bx^{-1}}{cx+dx^{-1}}$.

9. $\lim\limits_{x\doteq 0} \dfrac{ax+bx^{-1}}{cx+dx^{-1}}$.

10. $\lim\limits_{x=-\infty} \dfrac{\sqrt{1+x^2}}{x}$.

11. $\lim\limits_{x=\infty} \dfrac{x^2+x+1}{\sqrt{3+5x^2+4x^4}}$.

12. $\lim\limits_{x=\infty} \dfrac{\sqrt{1+x^2+x^4}}{x}$.

13. $\lim\limits_{x=\infty} \dfrac{x}{\sqrt{1+x^4}}$.

6. General Formulas of Differentiation.

THEOREM I. *The derivative of the product of a constant and a function is equal to the product of the constant into the derivative of the function:*

(I) $$D_x(cu) = cD_xu.$$

For, let $$y = cu.$$

Then $$y_0 = cu_0,$$

$$y_0 + \Delta y = c(u_0 + \Delta u),$$

hence $$\Delta y = c\Delta u,$$

$$\frac{\Delta y}{\Delta x} = c\frac{\Delta u}{\Delta x},$$

and $$\lim_{\Delta x\doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\doteq 0}\left(c\frac{\Delta u}{\Delta x}\right).$$

The limit of the left-hand side is $D_x y$. On the right, $\Delta u/\Delta x$ approaches $D_x u$ as its limit. Hence by § 5, Theorem II, the limit of the right-hand side is $cD_x u$, and we have

$$D_x(cu) = cD_x u, \qquad \text{q. e. d.}$$

THEOREM II. *The derivative of the sum of two functions is equal to the sum of their derivatives:*

$$\text{(II)} \qquad D_x(u+v) = D_x u + D_x v.$$

For, let
$$y = u + v.$$
Then
$$y_0 = u_0 + v_0,$$
$$y_0 + \Delta y = u_0 + \Delta u + v_0 + \Delta v,$$
hence
$$\Delta y = \Delta u + \Delta v,$$
and
$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

When Δx approaches 0, the first term on the right approaches $D_x u$ and the second $D_x v$. Hence by § 5, Theorem I, the whole right-hand side approaches $D_x u + D_x v$, and we have

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0}\left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}\right) = \lim_{\Delta x \doteq 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \doteq 0} \frac{\Delta v}{\Delta x},$$

or
$$D_x y = D_x u + D_x v, \qquad \text{q. e. d.}$$

COROLLARY. *The derivative of the sum of any number of functions is equal to the sum of their derivatives.*

If we have the sum of three functions, we can write

$$u + v + w = u + (v + w).$$

Hence
$$D_x(u + v + w) = D_x u + D_x(v + w)$$
$$= D_x u + D_x v + D_x w.$$

Next, we can consider the sum of four functions, and so on. Or we can extend the proof of Theorem II immediately to the sum of n functions.

Polynomials. We are now in a position to differentiate any polynomial. For example:

$$D_x(7x^4 - 5x^3 + x + 2)$$
$$= D_x(7x^4) + D_x(-5x^3) + D_x x + D_x 2$$
$$= 7 D_x x^4 - 5 D_x x^3 + 1 = 28x^3 - 15x^2 + 1.$$

EXERCISES

Differentiate the following functions:

1. $y = 2x^2 - 3x + 1.$ *Ans.* $D_x y = 4x - 3.$

2. $y = a + bx + cx^2.$ *Ans.* $D_x y = b + 2cx.$

3. $y = x^4 - 3x^3 + x - 1.$ *Ans.* $D_x y = 4x^3 - 9x^2 + 1.$

4. $y = a + bx + cx^2 + dx^3.$

5. $y = \dfrac{x^6 - 3x^4 - 2x + 1}{2}.$ *Ans.* $3x^5 - 6x^3 - 1.$

6. $f(x) = \dfrac{ax^2 + 2bx + c}{2h}.$ *Ans.* $\dfrac{ax + b}{h}.$

7. $\pi x^4 - 3\frac{3}{4}x^2 + \sqrt{3}.$ *Ans.* $4\pi x^3 - 7\frac{1}{2}x.$

8. Differentiate

(*a*) $v_0 t - 16t^2$ with respect to t;

(*b*) $a + bs + cs^2$ with respect to s;

(*c*) $.01ly^4 - 8.15my^2 - .9lm$ with respect to y.

9. Find the slope of the curve

$$4y = x^4 - 8x - 1$$

at the point $(1, -2)$. *Ans.* $-1.$

10. At what angle does the curve

$$8y = 4x - x^3$$

cut the negative axis of x?

11. At what angles do the curves $y = x^2$ and $y = x^3$ intersect?
Ans. 0° and 8° 7′.

12. At what angles do the curves $y = x^3 - 3x$ and $y = x$ intersect?
Ans. 26° 34′ and 38° 40′.

7. General Formulas of Differentiation, Continued.

THEOREM III. *The derivative of a product is given by the formula:*

$$\text{(III)} \qquad D_x(uv) = uD_xv + vD_xu.$$

Let
$$y = uv.$$
Then
$$y_0 = u_0v_0,$$
$$y_0 + \Delta y = (u_0 + \Delta u)(v_0 + \Delta v),$$
$$\Delta y = u_0\Delta v + v_0\Delta u + \Delta u\Delta v,$$
$$\frac{\Delta y}{\Delta x} = u_0\frac{\Delta v}{\Delta x} + v_0\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x},$$

and, by Theorem I, § 5:

$$\lim_{\Delta x \doteq 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0}\left(u_0\frac{\Delta v}{\Delta x}\right) + \lim_{\Delta x \doteq 0}\left(v_0\frac{\Delta u}{\Delta x}\right) + \lim_{\Delta x \doteq 0}\left(\Delta u\frac{\Delta v}{\Delta x}\right).$$

By Theorem II, § 5, the last limit has the value 0, since $\lim \Delta u = 0$ and $\lim(\Delta v/\Delta x) = D_xv$. The first two limits have the values u_0D_xv and v_0D_xu respectively.* Hence, dropping the subscripts, we have:

$$D_xy = uD_xv + vD_xu, \qquad \text{q. e. d.}$$

By a repeated application of this theorem the product of any number of functions can be differentiated. When more

* More strictly, the notation should read here, before the subscripts are dropped: $[D_xv]_{x=x_0}$, etc. Similarly in the proofs of Theorem I, II, and V.

than two factors are present, the formula is conveniently written in the form:

$$(1) \qquad \frac{D_x(uvw)}{uvw} = \frac{D_x u}{u} + \frac{D_x v}{v} + \frac{D_x w}{w}.$$

For a reason that will appear later, this is called the *logarithmic derivative* of uvw.

THEOREM IV. *The derivative of a quotient is given by the formula:**

$$(\text{IV}) \qquad D_x\left(\frac{u}{v}\right) = \frac{vD_x u - uD_x v}{v^2}.$$

Let
$$y = \frac{u}{v}.$$

Then
$$y_0 = \frac{u_0}{v_0}, \qquad y_0 + \Delta y = \frac{u_0 + \Delta u}{v_0 + \Delta v},$$

$$\Delta y = \frac{u_0 + \Delta u}{v_0 + \Delta v} - \frac{u_0}{v_0} = \frac{v_0 \Delta u - u_0 \Delta v}{v_0(v_0 + \Delta v)},$$

$$\frac{\Delta y}{\Delta x} = \frac{v_0 \dfrac{\Delta u}{\Delta x} - u_0 \dfrac{\Delta v}{\Delta x}}{v_0(v_0 + \Delta v)}.$$

By Theorem III of § 5 we have:

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \frac{\lim\limits_{\Delta x \doteq 0}\left(v_0 \dfrac{\Delta u}{\Delta x} - u_0 \dfrac{\Delta v}{\Delta x}\right)}{\lim\limits_{\Delta x \doteq 0}\left[v_0(v_0 + \Delta v)\right]}.$$

Applying Theorems I and II of § 5 and dropping the subscripts we obtain:

$$D_x y = \frac{vD_x u - uD_x v}{v^2}, \qquad \text{q. e. d.}$$

* The student may find it convenient to remember this formula by putting it into words: "The denominator into the derivative of the numerator, minus the numerator into the derivative of the denominator, over the square of the denominator."

Example 1. Let
$$y=\frac{2-3x}{1-2x}.$$

Then
$$D_xy=\frac{(1-2x)D_x(2-3x)-(2-3x)D_x(1-2x)}{(1-2x)^2}$$
$$=\frac{(1-2x)(-3)-(2-3x)(-2)}{(1-2x)^2}=\frac{1}{(1-2x)^2}.$$

Example 2. To prove that the theorem
$$D_xx^n=nx^{n-1}$$
is true when n is a negative integer, $n=-m$. Here
$$x^n=\frac{1}{x^m}.$$

Hence
$$D_xx^n=\frac{x^mD_x1-1D_xx^m}{x^{2m}}=-\frac{mx^{m-1}}{x^{2m}}=-mx^{-m-1}.$$

On replacing m in this last expression by its value, $-n$, the proof is complete.

EXERCISES

Differentiate the following functions:

1. $y=\dfrac{x}{1-x^2}$. *Ans.* $D_xy=\dfrac{1+x^2}{(1-x^2)^2}$.

2. $y=\dfrac{1}{1+x^2}$. *Ans.* $D_xy=\dfrac{-2x}{(1+x^2)^2}$.

3. $y=\dfrac{x^3}{1-x}$. *Ans.* $D_xy=\dfrac{3x^2-2x^3}{(1-x)^2}$.

4. $y=\dfrac{x^2}{1+x}$. *Ans.* $D_xy=\dfrac{2x+x^2}{(1+x)^2}$.

5. $s=\dfrac{1-t}{1+t}$. *Ans.* $D_ts=\dfrac{-2}{(1+t)^2}$.

6. $\dfrac{z^2+a^2}{z+a}$. *Ans.* $\dfrac{z^2+2az-a^2}{z^2+2az+a^2}$.

7. $\dfrac{2ay}{a^2 - y^2}$.

8. $\dfrac{ax + b}{x^2 + px + q}$.

9. $\dfrac{x^3 + a^3}{x + a}$.

10. $\dfrac{x^2 + a^2}{x^4 + a^4}$.

11. $\dfrac{a + bx + cx^2}{x}$. *Ans.* $c - \dfrac{a}{x^2}$.

12. $\dfrac{3 - 4x + x^3}{x^2}$.

13. $\dfrac{x^6 + 1}{x^3}$.

8. General Formulas of Differentiation, Concluded. *Composite Functions.*

Theorem V. *If u is expressed as a function of y and y in turn as a function of x:*

$$u = f(y), \qquad y = \phi(x),$$

then

$$D_x u = D_y u \cdot D_x y. \tag{V}$$

Here $\qquad y_0 = \phi(x_0), \qquad u_0 = f(y_0),$

$$y_0 + \Delta y = \phi(x_0 + \Delta x), \qquad u_0 + \Delta u = f(y_0 + \Delta y),$$

$$\Delta u = f(y_0 + \Delta y) - f(y_0),$$

$$\frac{\Delta u}{\Delta x} = \frac{f(y_0 + \Delta y) - f(y_0)}{\Delta y} \cdot \frac{\Delta y}{\Delta x}.$$

When Δx approaches 0, Δy also approaches 0, and hence the limit of the right-hand side is

$$\left(\lim_{\Delta y \doteq 0} \frac{f(y_0 + \Delta y) - f(y_0)}{\Delta y}\right)\left(\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}\right) = D_y f(y) D_x y.$$

The limit of the left-hand side is $D_x u$. Consequently

$$D_x u = D_y u \cdot D_x y, \qquad \text{q. e. d.}$$

This equation can also be written in the form:

$$D_x u = D_y f(y) D_x \phi(x). \tag{V'}$$

The truth of the theorem does not depend on the particular letters by which the variables are denoted. We may replace, for example, x by t and y by x:

$$D_t u = D_x u \, D_t x.$$

Dividing through by the second factor on the right, we thus obtain the formula:

$$(\mathrm{V}'') \qquad D_x u = \frac{D_t u}{D_t x}.$$

Example 1. In § 4 we differentiated the function $\sqrt{x}$, and we saw that other radicals can be differentiated in a similar manner. But each new differentiation required the evaluation of $\lim \Delta y/\Delta x$ by working through the details of a limiting process. Theorem V enables us to avoid such computations, as the following example will show.

To differentiate the function

$$u = \sqrt{a^2 - x^2}.$$

Let $$y = a^2 - x^2.$$

Then $$u = \sqrt{y},$$

and the differentiation thus comes directly under Theorem V, if we set

$$f(y) = \sqrt{y}, \qquad \phi(x) = a^2 - x^2.$$

Hence we have:

$$(1) \qquad D_x u = D_y \sqrt{y} \; D_x(a^2 - x^2).$$

Now, the formula $$D_x \sqrt{x} = \frac{1}{2\sqrt{x}}$$

does not mean that the independent variable must be denoted by the letter x. If the independent variable is y, the formula reads:

$$D_y \sqrt{y} = \frac{1}{2\sqrt{y}}.$$

Consequently (1) can be written in the form:

(2) $$D_x u = \frac{1}{2\sqrt{y}}(-2x) = -\frac{x}{\sqrt{a^2 - x^2}}.$$

We have, then, as the final result:

$$D_x\sqrt{a^2 - x^2} = \frac{-x}{\sqrt{a^2 - x^2}}.$$

Example 2. To differentiate the function

$$y = \frac{1}{(1-x)^3}.$$

Let $$z = 1 - x.$$

Then $$y = z^{-3}.$$

To apply Theorem V in the present case, the letters u and y must be replaced respectively by y and z. Thus Theorem V reads here:

$$D_x y = D_z y D_x z,$$

or $$D_x y = D_z z^{-3} D_x(1 - x).$$

Since Formula (8) of § 2 has been extended to negative integral values of n by § 7, Ex. 2, we have:

$$D_z z^{-3} = -3z^{-4}.$$

Hence $$D_x y = -3z^{-4}(-1) = \frac{3}{z^4},$$

or $$D_x \frac{1}{(1-x)^3} = \frac{3}{(1-x)^4}.$$

EXERCISES

Differentiate the following functions:

1. $y = \sqrt{a^2 + x^2}$. *Ans.* $\dfrac{x}{\sqrt{a^2 + x^2}}$.

2.* $y = \dfrac{1}{\sqrt{a^2 - x^2}}$. *Ans.* $-\dfrac{x}{\sqrt{(a^2 - x^2)^3}}$.

* Note that Formula (8) of § 2 has also been shown to hold for the case $n = -\frac{1}{2}$; § 4, Ex. 1.

3. $y = \sqrt{1 + x + x^2}$. *Ans.* $\dfrac{1 + 2x}{2\sqrt{1 + x + x^2}}$.

4. $y = \dfrac{1}{\sqrt{3 - 2x + 4x^2}}$. *Ans.* $\dfrac{1 - 2x}{\sqrt{3 - 2x + 4x^2}}$.

5. $u = \dfrac{x}{(1 - x)^3}$. *Ans.* $\dfrac{1 + 2x}{(1 - x)^4}$.

6. $u = \dfrac{x^2 + 1}{(2 - 3x)^2}$. *Ans.* $\dfrac{6 + 4x}{(2 - 3x)^3}$.

7. $y = \dfrac{x^2}{(1 + 2x)^4}$.

8. $y = \dfrac{(1 - x)^3}{(2 + x)^3}$.

9. $y = \left(\dfrac{x}{1 - x}\right)^4$.

10. $u = \dfrac{x^2}{1 - 2x + x^2}$.

11.* $u = x(1 - x)^4$. *Ans.* $(1 - 5x)(1 - x)^3$.

12. $u = x(a + bx)^n$. *Ans.* $[a + (n + 1)bx](a + bx)^{n-1}$.

13. $u = x^2(a + bx)^n$.

14. $u = x^3(1 - x)^4$.

15. $u = x\sqrt{a - x}$. *Ans.* $\dfrac{2a - 3x}{2\sqrt{a - x}}$.

16. $u = x^2\sqrt{a^2 - x^2}$.

17. $u = x\sqrt{1 + x + x^2}$.

18. $u = \dfrac{x}{\sqrt{a^2 - x^2}}$.

19. $u = \dfrac{x}{\sqrt{1 + x + x^2}}$.

20. $u = \sqrt{\dfrac{a + bx}{c + dx}}$.

21.† $u = \dfrac{1}{(a^2 - 2ax)^4}$.

22. $u = \dfrac{1}{(x^2 - 1)^2}$.

23. $u = \dfrac{1}{1 + x + x^2}$.

24. $u = \dfrac{x^3 - 3bx^2 + 3b^2x - b^3}{b - x}$.

25. $u = \dfrac{a + b}{(a + bx)^2}$.

* Use Theorem III.

† Do not use Theorem IV.

9. Differentiation of Implicit Algebraic Functions. When x and y are connected by such a relation as

$$x^2 + y^2 = a^2,$$

or
$$x^3 - 2xy + y^5 = 0,$$

or
$$xy \sin y = x + y \log x,$$

i.e. if y is given as a function of x by an equation,

$$F(x, y) = 0 \qquad \text{or} \qquad \Phi(x, y) = \Psi(x, y),$$

which must first be solved for y, then y is said to be an *implicit function* of x. If we solve the equation for y, thus obtaining the equation

$$y = f(x),$$

y thereby becomes an *explicit function* of x.

By an *algebraic function* of x is meant a function y which satisfies an equation of the form

$$G(x, y) = 0,$$

where $G(x, y)$ is an *irreducible* polynomial in x and y; *i.e.* a polynomial that cannot be factored and written as the product of two polynomials.

Thus the polynomials are algebraic functions; for if

$$y = a_0 + a_1 x + \cdots + a_n x^n = P(x),$$

then y satisfies the algebraic equation

$$G(x, y) = y - P(x) = 0.$$

Similarly, the fractions in x are algebraic functions; for if

$$y = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials having no common factor, then y satisfies the algebraic equation

$$G(x, y) = Q(x)y - P(x) = 0.$$

The polynomials and the fractions are also called *rational functions.* Thus,

$$\frac{ax+by}{x^2+y^2}$$

is a rational function of the two independent variables x and y.

Again, all roots of polynomials, as

$$y=\sqrt{1+x+x^3},$$

or such functions as

$$y=\sqrt[5]{\frac{x}{1-x}}+\sqrt{4-3x^2},$$

are algebraic, as is seen on freeing the equation from radicals and transposing. The converse, however, — namely, that every algebraic function can be expressed by means of rational functions and radicals, — is not true.

In order to differentiate an algebraic function, it is sufficient to differentiate the equation as it stands. Thus if

(1) $$x^2+y^2=a^2,$$

we have

(2) $$D_xx^2+D_xy^2=D_xa^2.$$

To find the value of the second term, apply Theorem V, § 8.

Thus $$D_xy^2=D_yy^2D_xy=2yD_xy.$$

This last factor, D_xy, is precisely the derivative we wish to find, and it is given by completing the differentiations indicated in (2):

$$2x+2yD_xy=0,$$

and solving this equation for D_xy:

$$D_xy=-\frac{x}{y}.$$

The final result is, of course, the same as if we had solved equation (1) for y:

$$y=\pm\sqrt{a^2-x^2},$$

and then differentiated:

$$D_xy = \pm\frac{-x}{\sqrt{a^2-x^2}} = -\frac{x}{y}.$$

In the case, however, of the equation

(3) $$x^3 - 2xy + y^5 = 0,$$

we cannot solve for y and obtain an explicit function expressed in terms of radicals. Nevertheless, the equation defines y as a perfectly definite function of x; for, on giving to x any special numerical value, as $x = 2$, we have an algebraic function for y, — here,

$$y^5 - 4y + 8 = 0,$$

and the roots of this equation can be computed to any degree of precision.

To find the derivative of this function, differentiate equation (3) as it stands with respect to x:

(4) $$D_xx^3 - 2D_x(xy) + D_xy^5 = 0.$$

The second term in this last equation can be evaluated by Theorem III of § 7:

$$D_x(xy) = xD_xy + y,$$

where D_xy denotes the derivative we wish to find.

To the evaluation of the third term in (4) Theorem V of § 8 applies:

$$D_xy^5 = 5y^4D_xy.$$

Hence

$$3x^2 - 2xD_xy - 2y + 5y^4D_xy = 0.$$

Solving this equation for D_xy, we have as the final result:

$$D_xy = \frac{2y-3x^2}{5y^4-2x}.$$

Thus, for example, the curve is seen to go through the point (1, 1), and its slope there is

$$(D_xy)_{(1,\,1)} = -\tfrac{1}{3}.$$

The differentiation of implicit functions as set forth in the above examples is based on the assumptions *a*) that the given equation defines y as a function of x; *b*) that this function has a derivative. The proof of these assumptions belongs to a more advanced stage of analysis. In the case, however, of the equations we meet in practice, — for example, such equations as come from a problem in geometry or physics, — the conditions for the existence of a solution and of its derivative are fulfilled, and we shall take it for granted henceforth that this is true of the implicit functions we meet.

Derivative of x^n, n Fractional. We are now in a position to prove the theorem

$$D_x x^n = nx^{n-1}$$

for the case that n is a fraction. Let

$$n = \frac{p}{q},$$

where p, q are whole numbers which are prime to each other. Let

$$y = x^{\frac{p}{q}}.$$

Then

$$y^q = x^p.$$

Differentiating each side of this equation with respect to x, we have:

$$D_x y^q = D_x x^p,$$

and since, by Theorem V, § 8,

$$D_x y^q = D_y y^q D_x y = qy^{q-1} D_x y,$$

it follows that

$$qy^{q-1} D_x y = px^{p-1},$$

or

$$D_x y = \frac{p}{q} \frac{x^{p-1}}{y^{q-1}}.$$

This last denominator has the value

$$(x^{\frac{p}{q}})^{q-1} = x^{p-\frac{p}{q}}.$$

Hence $$\frac{x^{p-1}}{y^{q-1}} = \frac{x^{p-1}}{x^{p-\frac{p}{q}}} = x^{\frac{p}{q}-1}.$$

We see, then, that

$$D_x y = \frac{p}{q} x^{\frac{p}{q}-1} = nx^{n-1}, \qquad \text{q. e. d.}$$

If, finally, n is a negative fraction, $n = -m$, the proof can be given precisely as was done in § 7, Ex. 2. Thus the theorem

$$D_x x^n = nx^{n-1}$$

is now established for all commensurable values of n.

The theorem is true even when n is irrational, *e.g.* $n = \pi$ or $\sqrt{2}$; the proof depends on the logarithmic function and will be given when that function has been differentiated.

Example. Differentiate the function

$$y = \sqrt[3]{a^3 - x^3}.$$

Apply Theorem V, § 8, setting

$$z = a^3 - x^3.$$

Then $$D_x y = D_x z^{\frac{1}{3}} = D_z z^{\frac{1}{3}} D_x z = \tfrac{1}{3} z^{-\frac{2}{3}}(-3x^2).$$

Hence $$D_x \sqrt[3]{a^3 - x^3} = \frac{-x^2}{\sqrt[3]{(a^3 - x^3)^2}}.$$

EXERCISES

1. If $$2x^3 - 3x^2y + 4xy + 6y^3 = 0,$$
find $D_x y$. *Ans.* $D_x y = \dfrac{6x^2 - 6xy + 4y}{3x^2 + 4x + 18y^2}$.

2. If $$y^4 - 2xy^2 = x^4,$$
find $D_x y$.

3. Show that the curve

$$x^4 - 2xy^2 + y^3 + 3x - 3y = 0$$

cuts the axis of x at the origin at an angle of 45°.

4. Plot the curve $x^4 + y^4 = 81,$

taking 1 cm. as the unit. Show that this curve is cut orthogonally by the bisectors of the angles made by the coordinate axes.

Differentiate the following functions:

5. $u = \sqrt[5]{1 - x}.$ *Ans.* $\dfrac{-1}{5\sqrt[5]{(1-x)^4}}.$

6. $u = \sqrt[3]{a^2 - 2\,ax + x^2}.$ *Ans.* $\dfrac{-2}{3\sqrt[3]{a-x}}.$

7. $u = \sqrt[5]{c^3 - 3\,c^2x + 3\,cx^2 - x^3}.$ *Ans.* $-\dfrac{3}{5\sqrt[5]{c^2 - 2\,cx + x^2}}.$

8. $u = \sqrt[3]{\dfrac{x}{1-x}}.$ *Ans.* $\dfrac{1}{3\sqrt[3]{x^2(1-x)^4}}.$

9. $u = x\sqrt[4]{a + bx + cx^2}.$

10. $u = \dfrac{\sqrt[4]{a^4 + x^4}}{x}.$

11. $\dfrac{\sqrt{a-x} + \sqrt{a+x}}{\sqrt{a-x} - \sqrt{a+x}}.$ *Ans.* $\dfrac{a^2 + a\sqrt{a^2 - x^2}}{x^2\sqrt{a^2 - x^2}}.$

12. $y = \sqrt[3]{ax^2}.$

13. $r = \sqrt{a\theta}.$

14. $u = \dfrac{1 - x^{-\frac{1}{2}}}{x^{\frac{1}{5}}}.$ *Ans.* $D_x u = \dfrac{7 - 2\sqrt{x}}{10\sqrt[10]{x^{17}}}.$

15. $y = \dfrac{1 + x^2}{\sqrt[3]{x}}.$

16. $u = x\sqrt{2\,x}.$

17. $v = 4\sqrt[3]{t^2} + \dfrac{3}{\sqrt{t}} - 1.$

18. $(y^2 + 1)\sqrt{y^3 - y}.$ *Ans.* $\dfrac{7\,y^4 - 2\,y^2 - 1}{2\sqrt{y^3 - y}}.$

19. $\dfrac{(s^2 - a^2)^{\frac{3}{2}}}{s^3}.$ *Ans.* $\dfrac{3\,a^2\sqrt{s^2 - a^2}}{s^4}.$

20. $\dfrac{a - x}{\sqrt{2ax - x^2}}$.

21. $\sqrt[3]{\dfrac{1 - x}{(1 + x)^2}}$.

22. $v = x(a^2 - x^2)^{\frac{3}{7}}$.

23. $u = (b - t)\sqrt{b + t}$.

24. Find the slope of the curve $y = x^{\frac{1}{5}}$ in the point whose abscissa is 2. *Ans.* $\tan\tau = .115$.

25. If $pv^{1.4} = c$, find $D_v p$.

26. If $y\sqrt{x} = 1 + x$, find $D_x y$. *Ans.* $\dfrac{x - 1}{2x\sqrt{x}}$.

27. Differentiate y in two ways, where

$$xy + 4y = 3x,$$

and show that the results agree.

28. The same, when $y^2 = 2mx$.

29. Show that the curves

$$3y = 2x + x^4y^3, \qquad 2y + 3x + y^5 = x^3y,$$

intersect at right angles at the origin.

30. Find the angle at which the curves

$$2x = x^4 - xy + x^5, \qquad x^4 + y^4 + 5x = 7y,$$

intersect at the origin. *Ans.* $\tan\phi = 1.4$.

CHAPTER III

APPLICATIONS

1. Tangents and Normals. By the tangent line, or simply the *tangent*, to a curve at any one of its points, P, is meant the straight line through P, whose slope is the same as that of the curve at that point.

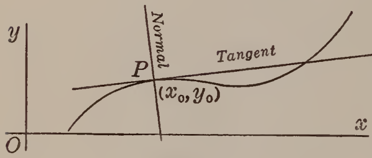

FIG. 9

Let the coordinates of P be denoted by (x_0, y_0). Now, the equation of the straight line through P, whose slope is λ, is

$$y - y_0 = \lambda(x - x_0).$$

On the other hand, the slope of the curve at any point is $D_x y$. If we denote the value of this slope at (x_0, y_0) by $(D_x y)_0$, this will be the desired value of λ:

$$\lambda = (D_x y)_0.$$

Hence the equation of the tangent to the curve

$$y = f(x) \qquad \text{or} \qquad F(x, y) = 0$$

at the point (x_0, y_0) is

(1) $$y - y_0 = (D_x y)_0 (x - x_0).$$

Since the *normal* is perpendicular to the tangent, its slope, λ', is the negative reciprocal of the slope of that line, or

$$\lambda' = -\frac{1}{(D_x y)_0}.$$

Hence the equation of the normal to the curve at (x_0, y_0) is

$$(2)\quad y - y_0 = -\frac{1}{(D_xy)_0}(x - x_0) \quad \text{or} \quad x - x_0 + (D_xy)_0 \cdot (y - y_0) = 0.$$

Example 1. To find the equation of the tangent to the curve

$$y = x^3$$

in the point $x = \frac{1}{2}$, $y = \frac{1}{8}$. Here

$$D_xy = 3x^2, \qquad (D_xy)_0 = [3x^2]_{x=\frac{1}{2}} = \tfrac{3}{4}.$$

Hence the equation of the tangent is

$$y - \tfrac{1}{8} = \tfrac{3}{4}(x - \tfrac{1}{2}) \qquad \text{or} \qquad 3x - 4y - 1 = 0.$$

Example 2. Let the curve be an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Differentiating the equation as it stands, we get:

$$\frac{2x}{a^2} + \frac{2y}{b^2} D_xy = 0, \qquad D_xy = -\frac{b^2x}{a^2y}.$$

Hence the equation of the tangent is

$$y - y_0 = -\frac{b^2x_0}{a^2y_0}(x - x_0).$$

This can be transformed as follows:

$$a^2y_0y - a^2y_0^2 = -b^2x_0x + b^2x_0^2,$$

$$b^2x_0x + a^2y_0y = a^2y_0^2 + b^2x_0^2 = a^2b^2,$$

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1.$$

EXERCISES

1. Find the equation of the tangent of the curve

$$y = x^3 - x$$

at the origin; at the point where it crosses the positive axis of x. *Ans.* $x + y = 0$; $2x - y - 2 = 0$.

2. Find the equation of the tangent and the normal of the circle

$$x^2 + y^2 = 4$$

at the point $(1, \sqrt{3})$ and check your answer.

3. Show that the equation of the tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1.$$

4. Show that the equation of the tangent of the parabola

$$y^2 = 2mx$$

at the point (x_0, y_0) is

$$y_0 y = m(x + x_0).$$

5. Show that the equation of the tangent of the parabola

$$y^2 = m^2 - 2mx$$

at the point (x_0, y_0) is

$$y_0 y = m^2 - m(x + x_0).$$

6. Show that the equation of the tangent of the equilateral hyperbola

$$xy = a^2$$

at the point (x_0, y_0) is

$$y_0 x + x_0 y = 2a^2.$$

7. Find the equation of the tangent to the curve

$$x^3 + y^3 = a^2(x - y)$$

at the origin. *Ans.* $x = y$.

8. Show that the area of the triangle formed by the coordinate axes and the tangent of the hyperbola

$$xy = a^2$$

at any point is constant.

9. Find the equation of the tangent and the normal of the curve

$$x^5 = a^3y^2$$

in the point distinct from the origin in which it is cut by the bisector of the positive coordinate axes.

10. Show that the portion of the tangent of the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

at any point, intercepted between the coordinate axes, is constant.

11. The parabola $y^2 = 2ax$ cuts the curve

$$x^3 - 3axy + y^3 = 0$$

at the origin and at one other point. Write down the equation of the tangent of each curve in the latter point.

12. Show that the curves of the preceding question intersect in the second point at an angle of 32° 12′.

2. Maxima and Minima. *Problem.* From a piece of tin 3 ft. square a box is to be made by cutting out equal squares from the four corners and bending up the sides. Determine the dimensions of the box of this description which will hold the most.

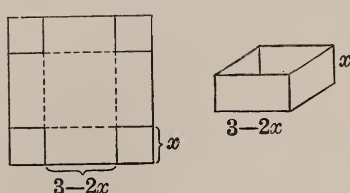

Fig. 10

Solution. Let x be the length of the side of the square removed; then the dimensions of the box are as indicated in the diagrams. Denoting the cubical content of the box by u, we have:

1) $$u = x(3-2x)^2,$$

or

2) $$u = 9x - 12x^2 + 4x^3.$$

The problem is, then, to find the value of x which makes u as large as possible, x being restricted from the nature of the case to being positive and less than $\frac{3}{2}$:

3) $$0 < x < \tfrac{3}{2}.$$

The problem can be treated graphically by plotting the curve 1). We wish to find the highest point on this curve. It appears to be the point for which $x = \frac{1}{2}$, $u = 2$, since other values of x which have been tried lead to smaller values of u.

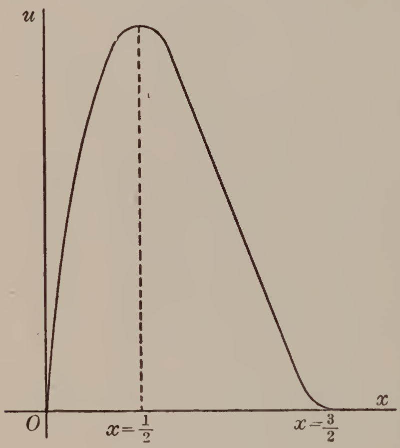

FIG. 11

The foregoing method has the advantage that it is direct, for it assumes no knowledge of mathematics beyond curve plotting. It has the disadvantage that curve plotting, even in the simplest cases, is laborious; and, furthermore, we have not really proved that $x = \frac{1}{2}$ is the best value. We have merely failed to find a better one.

The Calculus supplies a means of meeting both the difficulties mentioned, and yielding a solution with the greatest ease.

The problem is to find the *highest* point on the curve. At this point, the tangent of the curve is evidently parallel to the axis of x. Consequently, the slope of the tangent, *i.e.* $\tan\tau = D_x u$, must have the value 0 here:

$$D_x u = 0.$$

All we need do, therefore, is to compute $D_x u$, most conveniently from equation 2), and set the result equal to 0:

$$D_x u = 9 - 24x + 12x^2 = 0.$$

On solving this quadratic equation for x, we find two roots,

$$x = \tfrac{1}{2}, \quad \tfrac{3}{2}.$$

Only one of these, however, lies within the range 3) of possible values for x, namely, the value $x = \frac{1}{2}$, and hence this is the required value.

EXERCISES

1. Work the foregoing problem for the case that the tin is a rectangle 1 by 2 ft.

Plot accurately the graph, taking 10 cm. as the unit, and determine in this way what appears to be the best value for x, correct to one eighth of an inch.

Solve the problem by the Calculus, and show that the best value for x is .21132 ft., or 2.5359 in.

2. A farmer wishes to fence off a rectangular pasture along a straight river, one side of the pasture being formed by the river and requiring no fence. He has barbed wire enough to build a fence 1000 ft. long. What is the area of the largest pasture of the above description which he can fence off?

3. Show that, of all rectangles having a given perimeter, the square has the largest area.

4. Show that, of all rectangles having a given area, the square has the least perimeter.

5. Each side of a shelter tent is a rectangle 6×8 ft. How must the tent be pitched so as to afford the largest amount of room inside? The ends are to be open.

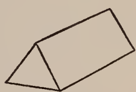

FIG. 12

Ans. The angle along the ridge-pole must be a right angle.

6. Divide the number 12 into two parts such that the sum of their squares may be as small as possible.

(What is meant is such a division as this: one part might be 4, and then the other would be 8. The sum of the squares would here be $16 + 64 = 80$.)

7. Divide the number 8 into two such parts that the sum of the cube of one part and twice the cube of the other may be as small as possible.

8. Divide the number 9 into two such parts that the product of one part by the square of the other may be as large as possible.

9. Divide the number 8 into two such parts that the product of one part by the cube of the other may be as large as possible.

10. At noon, one ship, which is steaming east at the rate of 20 miles an hour, is due south of a second ship steaming south at 16 miles an hour, the distance between them being 82 miles. If both ships hold their courses, show that they will be nearest to each other at 2 P.M.

11. If, in the preceding problem, the second ship lies to from noon till one o'clock, and then proceeds on her southerly course at 16 miles an hour, when will the ships be nearest to each other?

12. Find the least value of the function

$$y = x^2 + 6x + 10.$$ *Ans.* 1.

13. What is the greatest value of the function

$$y = 3x - x^3$$

for positive values of x?

14. For what value of x does the function

$$\frac{12\sqrt{x}}{1 + 4x}$$

attain its greatest value? *Ans.* $x = \frac{1}{4}$.

15. At what point of the interval $a < x < b$, a being positive, does the function

$$\frac{x}{(x-a)(b-x)}$$

attain its least value? *Ans.* $x = \sqrt{ab}$.

16. Find the most advantageous length for a lever, by means of which to raise a weight of 490 lb. (see Fig. 13), if the distance of the weight from the fulcrum is 1 ft. and the lever weighs 5 lb. to the foot.

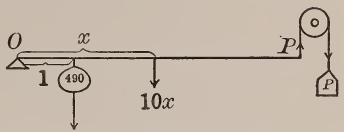

FIG. 13

3. Continuation: Auxiliary Variables. It frequently, — in fact, usually, — happens that it is more convenient to formulate a problem if more variables are introduced at the outset than are ultimately needed. The following examples will serve to illustrate the method.

Example 1. Let it be required to find the rectangle of greatest area which can be inscribed in a given circle.

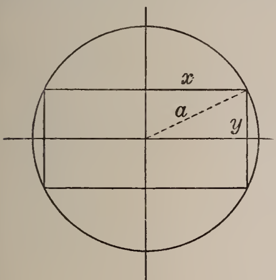

FIG. 14

It is evident that the area of the rectangle will be small when its altitude is small and also when its base is short. Hence the area will be largest for some intermediate shape.

Let u denote the area of the rectangle. Then

(1) $$u = 4xy.$$

But x and y cannot both be chosen arbitrarily, for then the rectangle will not in general be inscriptible in the given circle. In fact, it is clear from the Pythagorean Theorem that x and y must satisfy the relation:

(2) $$x^2 + y^2 = a^2.$$

We could now eliminate y between equations (1) and (2), thus obtaining u in terms of x alone; and it is, indeed, im-

portant to think of this elimination as performed, for there is only *one* independent variable in the problem. The graph of u, regarded as a function of x, starts at the origin, rises as x increases, but finally comes back to the axis of x again when $x = a$. All this we read off, either from the meaning of u and x in the problem or from equations (1) and (2).

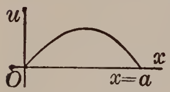

FIG. 15

It is better, however, in practice not to eliminate y, but to differentiate equations (1) and (2) with respect to x as they stand, and then set $D_x u = 0$. Thus from (1),

$$D_x u = 4(y + xD_x y) = 0,$$

and from (2),
$$2x + 2yD_x y = 0.$$

From the second of these equations we see that

$$D_x y = -\frac{x}{y}.$$

Substituting for $D_x y$ this value in the first equation, we get:

$$y - \frac{x^2}{y} = 0 \qquad \text{or} \qquad y^2 = x^2.$$

Since x and y are both positive numbers, it follows that

$$y = x.$$

Hence the maximum rectangle is a square.

EXERCISES

1. Work the same problem for an ellipse, instead of a circle.

2. Work the problem for the case of a variable rectangle inscribed in a fixed equilateral triangle.

Example 2. To find the most economical dimensions for a tin dipper, to hold a pint.

Here, the amount of tin required is to be as small as possible, the content of the dipper being given. Let u denote the surface, measured in square inches. Then

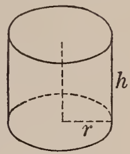

FIG. 16

a) $$u = 2\pi rh + \pi r^2.$$

But r and h cannot both be chosen arbitrarily, for then the dipper would not in general hold a pint. If V denotes the given volume, measured in cubic inches, then, since this volume can also be expressed as $\pi r^2 h$, we have

b) $$\pi r^2 h = V.$$

Differentiating equation *a*) with respect to r and setting $D_r u = 0$, we have:

$$D_r u = \pi\{2h + 2rD_r h + 2r\} = 0,$$

or

c) $$h + rD_r h + r = 0.$$

Differentiating *b*) we get:

d) $$\pi\{2rh + r^2 D_r h\} = 0.$$

Now, r cannot $= 0$ in this problem, and so we may divide this last equation through by r, as well as by π:

e) $$2h + rD_r h = 0.$$

It remains to eliminate $D_r h$ between equations *c*) and *e*). From *e*),

$$D_r h = -\frac{2h}{r}.$$

Substituting this value of $D_r h$ in *c*), we find:

f) $$h - 2h + r = 0, \qquad \text{or} \qquad r = h.$$

Hence the depth of the dipper must just equal its radius.

Discussion. Just what have we done here? The steps we have taken are suggested clearly enough by the solution of

Example 1. We have chosen one of the two variables, r and h, as the independent variable (here, r); differentiated the function u, which is to be made a minimum, with respect to r, and set $D_r u = 0$. Then we differentiated the second equation b), likewise with respect to r, eliminated $D_r u$, and solved. But what does it all mean? What is behind it all?

Just this: the quantity u, in the nature of the case, is a function of r. For, when to r is given *any* positive value, a dipper can be constructed which will fulfill the requirements. Now, if r is very large, we shall have a shallow pan, and evidently the amount of tin required to make it will be large; — *i.e.* u will also have a large value.

But what if r is small? We shall then have a high cylinder of minute cross sections, *i.e.* a pipe. Is it clear that u, the surface, will be large in this case, too? I fear not, for it is purely a relative question as to how high such a pipe must be to hold a pint, and I see no way of guessing intelligently. By means of equation b), however, we see that

$$h = \frac{V}{\pi r^2},$$

and if we substitute this value in a), we get

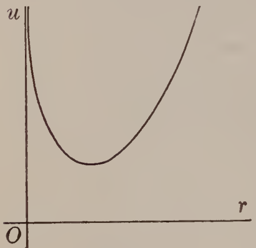

Fig. 17

$$u = 2\pi r \frac{V}{\pi r^2} + \pi r^2 = \frac{2V}{r} + \pi r^2.$$

From this last formula it is clear that, when r is small, u actually is large.

The graph of u, regarded as a function of r, is therefore in character as shown by the accompanying figure. It is a continuous curve lying above the axis of r, very high when r is small, and also very high when r is large. It has, therefore, a lowest point, and for this value of r, the area u of the dipper will be least. But at this lowest point the slope of the curve, $D_r u$, has the value 0. Thus we see, first, that we have a genu-

ine minimum problem;—there *is* actually a dipper of smallest area. Secondly, equations c) and d) must hold, and since from these equations it follows by elimination that $r = h$, there is *only one* such dipper, and its radius is equal to its altitude. The problem is, then, completely solved.

We inquired merely for the *shape* of the dipper. If the *size* had been asked for, too, it could be found by solving b) and f) for r and h, and expressing V in cubic inches:

$$V = \tfrac{231}{8} = 28.875,$$

$$\pi r^3 = 28.87, \qquad r = 2.095.$$

It can happen in practice that a function attains its greatest or its least value at the end of the interval. In that case, the derivative does not have to vanish. Usually, the facts are patent, and so no special investigation is needed. But it is necessary to assure oneself that a given problem which looks like one of the above does not come under this head, and this is done, as in the cases discussed in the text, by showing that near the ends of the interval the values of the function are larger, for a minimum problem, than for values well within the interval.

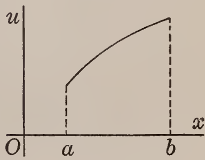

FIG. 18

EXERCISE

Discuss in a similar manner the best shape for a tomato can which is to hold a quart. Here, the tin for the top must also be figured in. Show that the height of such a can should be equal to the diameter of the base. As to the size of the can, its height should be 4.19 in.

A General Remark. It might seem as if the method used in the solution of the above problems were likely to be insecure, since the graph of such a function u might, in the very next problem, look like the accompanying figure. In such a case, there would be several values of x, for each of which $D_x u = 0$,

and we should not know which one to take. Curiously enough, this case does not arise in practice, — at least, I have never come across a physical problem which led to this difficulty. In problems like the above, there must be *at least* one x for which $D_x u = 0$; and when we solve a given problem, we actually find *only one* x which fulfills the condition. Thus there is no ambiguity.

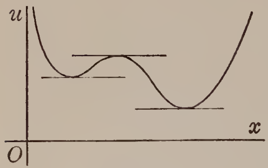

FIG. 19

EXERCISES

1. A 300-gallon tank is to be built with a square base and vertical sides, and is to be lined with copper. Find the most economical proportions.

Ans. The length and breadth must each be double the height.

2. Find the cylinder of greatest volume which can be inscribed in a given cone of revolution.

Ans. Its altitude is one-third that of the cone.

3. What is the cylinder of greatest convex surface that can be inscribed in the same cone?

Ans. Its altitude is half that of the cone.

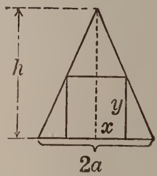

FIG. 20

4. Of all the cones which can be inscribed in a given sphere, find the one whose lateral area is greatest.

Ans. Its altitude exceeds the radius of the sphere by $33\frac{1}{3}\%$ of that radius.

5. Find the volume of the greatest cone of revolution which can be inscribed in a given sphere.

6. If the top and bottom of the tomato can considered in the Exercise of the text are cut from sheets of tin so that a regular hexagon is used up each time, the waste being a total loss, what will then be the most economical proportions for the can?

7. If the strength of a beam is proportional to its breadth and to the square of its depth, find the shape of the strongest beam that can be cut from a circular log.

Ans. The ratio of depth to breadth is $\sqrt{2}$.

8. Assuming that the stiffness of a beam is proportional to its breadth and to the cube of its depth, find the dimensions of the stiffest beam that can be sawed from a log one foot in diameter.

9. What is the shortest distance from the point (10, 0) to the parabola

$$y^2 = 4x?$$

10. What points of the curve

$$y^2 = x^3$$

are nearest (4, 0)?

11. A trough is to be made of a long rectangular-shaped piece of copper by bending up the edges so as to give a rectangular cross-section. How deep should it be made, in order that its carrying capacity may be as great as possible?

12. Assuming the density of water to be given from 0° to 30° C. by the formula

$$\rho = \rho_0(1 + \alpha t + \beta t^2 + \gamma t^3),$$

where ρ_0 denotes the density at freezing, t the temperature, and

$$\alpha = 5.30 \times 10^{-5}, \qquad \beta = -6.53 \times 10^{-6}, \qquad \gamma = 1.4 \times 10^{-8},$$

show that the maximum density occurs at $t = 4.08°$.

13. Tangents are drawn to the arc of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which lies in the first quadrant. Which one of them cuts off from that quadrant the triangle of smallest area?

14. Work the same problem for the parabola

$$y^2 = a^2 - 4ax.$$

15. Show that, of all circular sectors having the same perimeter, that one has the largest area for which the sum of the two straight sides is equal to the curved side.

4. Increasing and Decreasing Functions. The Calculus affords a simple means of determining whether a function is increasing or decreasing as the independent variable increases. Since the slope of the graph is given by D_xy, we see that when D_xy is positive, y increases as x increases, but when D_xy is negative, y decreases as x increases. Figure 21 shows the graph in general when D_xy is positive.

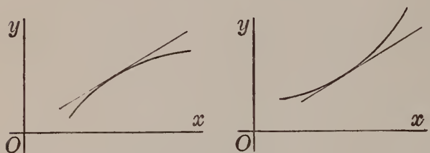

FIG. 21

In each figure both x and y have been taken as positive. But what is said above in the text is equally true when one or both of these variables are negative; for the words *increase* and *decrease* as here used mean *algebraic*, not numerical, increase or decrease. Thus if the temperature is ten degrees below zero (*i.e.* $-10°$) and it changes to eight below ($-8°$), we say the temperature has risen. If we measure the time t, in hours from noon, then 10 A.M. will correspond to $t = -2$. Let u denote the temperature, measured in degrees. Then a temperature chart for 24 hours from midnight to midnight might look like the accompanying figure. At any instant, $t = t'$, for which the slope of the curve, D_tu, is positive, the temperature is rising, no matter whether the thermometer is above zero or below, and no matter whether t is positive or negative; and similarly, when D_tu is negative, the temperature is falling.

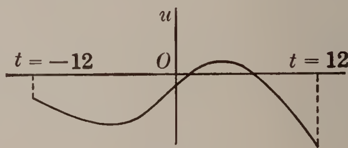

FIG. 22

Again, suppose the amount of business a department store

does in a year, as represented by the net receipts each day, be plotted as a curve (y = receipts, measured in dollars; x = time, measured in days), the curve being smoothed in the usual way. Then a point of the curve at which the derivative is positive (*i.e.* $D_x y > 0$) indicates that, at that time, the business of the firm was increasing; whereas a point at which $D_x y < 0$ means that the business was falling off.

We can state the result in the form of a general theorem, the proof of which is given by inspection of the figure (Fig. 21) and the other forms of the figure, brought out in the above discussion.

THEOREM: *When x increases, then*

(*a*) *if* $D_x y > 0$, *y increases;*

(*b*) *if* $D_x y < 0$, *y decreases.*

Application. As an application consider the condition that a curve $y = f(x)$ have its concave side turned upward, as in Fig. 23. The slope of the curve is a function of x:

$$\tan \tau = \phi(x).$$

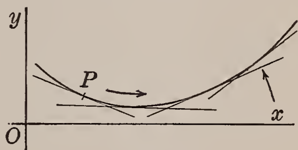

FIG. 23

For, when x is given, a point of the curve, and hence also the slope of the curve at this point, is determined. Consider the tangent line at a variable point P. If we think of P as tracing out the curve and carrying the tangent along with it, the tangent will turn in the counter clock-wise sense, the slope thus increasing algebraically as x increases, whenever the curve is concave upward. And conversely, if the slope increases as x increases, the tangent will turn in the counter clock-wise sense and the curve will be concave upward. Now by the above theorem, when

$$D_x \tan \tau > 0,$$

$\tan \tau$ increases as x increases. Hence the curve is concave upward, when $D_x \tan \tau$ is positive; and conversely.

The derivative $D_x \tan\tau$ is the derivative of the derivative of y. This is called the *second derivative of* y, and is denoted as follows:

$$D_x(D_x y) = D_x^2 y$$

(read: "$D\ x$ second of y").*

The test for the curve's being concave downward is obtained in a similar manner, and thus we are led to the following important theorem.

TEST FOR A CURVE'S BEING CONCAVE UPWARD, ETC. *The curve*

$$y = f(x)$$

is CONCAVE UPWARD *when* $D_x^2 y > 0$;

CONCAVE DOWNWARD *when* $D_x^2 y < 0$.

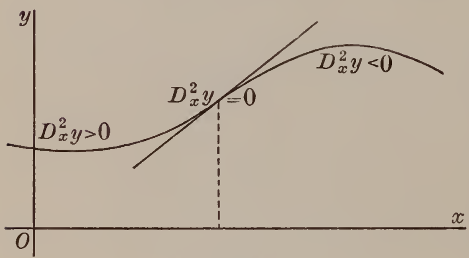

FIG. 24

A point at which the curve changes from being concave upward and becomes concave downward (or *vice versa*) is called a *point of inflection.* Since $D_x^2 y$ changes sign at such a point, this function will necessarily, if continuous, vanish there. Hence:

A necessary condition for a point of inflection is that

$$D_x^2 y = 0.$$

Example. Consider the curve

$$y = x^3 - 3x.$$

* The derivative of the second derivative, $D_x(D_x^2 y)$, is called the *third derivative* and is written $D_x^3 y$, and so on.

Its slope at any point is given by the equation

$$D_x y = 3x^2 - 3.$$

The second derivative of y with respect to x has the value

$$D_x^2 y = 6x.$$

Thus we see that this curve is concave upward for all positive values of x, and concave downward for all negative values. In character it is as shown in the accompanying figure.

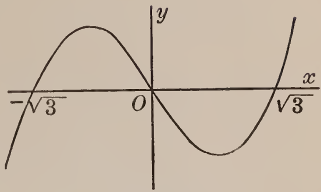

FIG. 25

EXERCISES

For what values of x are the following functions increasing? For what values decreasing?

1. $$y = 4 - 2x^2.$$

2. $$y = x^2 - 2x + 3.$$

Ans. Increasing, when $x > 1$; decreasing, when $x < 1$.

3. $$y = 5 + 12x - x^2.$$

4. $$y = x^3 - 27x + 7.$$

Ans. Increasing, when $x > 3$, and when $x < -3$; decreasing, when $-3 < x < 3$.

5. $y = 5 + 6x - x^3$. 6. $y = x - x^5$.

7. $$y = x^3 - 9x^2 + 12x - 1.$$

In what intervals are the following curves concave upward; in what, downward?

8. $$y = x^3 - 3x^2 + 7x - 5.$$

Ans. Concave upward, when $x > 1$; concave downward, when $x < 1$.

9. $y = 15 + 8x + 3x^2 - x^3$. 10. $y = x^3 - 6x^2 - x - 1$.

11. $y = 3 - 9x + 24x^2 - 4x^3$. 12. $y = 2x^3 - x^4$.

13. $y = x^4 - 4x^3 - 6x + 11$. 14. $y = -121x + 7x^3 - x^7$.

15. $$y = 13 + 23x - 24x^2 + 12x^3 - x^4.$$

5. Curve Tracing. In the early work of plotting curves from their equations the only way we had of finding out what the graph of a function looked like was by computing a large number of its points. We are now in possession of powerful methods for determining the character of the graph with scarcely any computation. For, first, we can find the slope of the curve at any point; and, secondly, we can determine in what intervals the curve is concave upward, in what concave downward.*

Example. Let it be required to plot the curve

(1) $$3y = x^3 - 3x^2 + 1.$$

a) Determine first its slope at any point:

(2) $$3D_xy = 3x^2 - 6x, \qquad D_xy = x^2 - 2x.$$

* There are two great applications of the graphical representation of a function. One is *quantitative*, the other, *qualitative*. By the first I mean the use of the graph as a *table*, for actual computation. Thus in the use of logarithms it is desirable to have a graph of the function $y = \log_{10} x$ drawn accurately for values of x between 1 and 10; for by means of such a graph the student can read off the logarithms he is using, correct to two or three significant figures, and so obtain a check on his numerical work.

There is, however, a second large and important class of problems, in which the *character* of a function is the important thing, a minute determination of its values being in general irrelevant.

A case in point is the determination of the number of roots of an algebraic equation, *e.g.*
$$x^3 - x^2 - 4x + 1 = 0,$$
Here, we plot the curve
$$y = x^3 - x^2 - 4x + 1$$
and inquire where it cuts the axis of x. For this purpose it is altogether adequate to know the character of the curve, and for treating this problem the methods of the present paragraph yield a powerful instrument.

It is always useful to know the points at which the tangent to the curve is parallel to the axis of x. These are obtained by setting $D_x y = 0$ and solving. Thus we get from (2) the equation:

$$x^2 - 2x = 0.$$

The roots of this equation are

$$x = 0 \qquad \text{and} \qquad x = 2$$

Now determine accurately the points having these abscissas, plot them, and draw the tangents there:

$$y\big|_{x=0} = \tfrac{1}{3}; \qquad y\big|_{x=2} = -1.$$

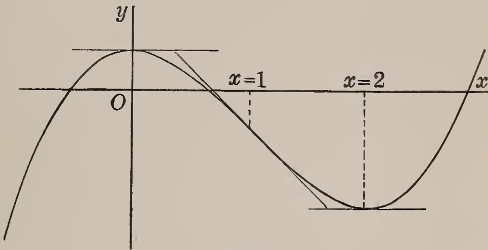

FIG. 26

We do not yet know whether the curve lies above its tangent in one of these points, or below its tangent; it might even cross its tangent, for the point might be a point of inflection. These questions will all be answered by aid of the second derivative.

b) Compute the second derivative:

$$D_x^2 y = 2x - 2 = 2(x - 1).$$

We see that it is positive when x is greater than 1 and negative when x is less than 1:

$$D_x^2 y > 0 \quad \text{when} \quad 1 < x;$$
$$D_x^2 y < 0 \quad \text{when} \quad x < 1.$$
$$D_x^2 y = 0 \quad \text{when} \quad x = 1.$$

Hence the curve has a point of inflection when $x = 1$. This is a most important point on the curve. We will compute its

coordinates accurately, determine the slope of the curve there, and draw accurately the tangent there.

$$y\,|_{x=1} = -\tfrac{1}{3}; \qquad D_x y\,|_{x=1} = -1.$$

This is the last of the important tangents which we need to draw. Since the curve is concave upward to the right of the line $x = 1$, and concave downward to the left of that line, it must be in character as indicated. We see, then, that it cuts the axis of x between 0 and 1, and again to the right of the point $x = 1$; and it cuts that axis a third time to the left of the origin.

These last two points can be located more accurately by computing the function for a few simple values of x.

$$y\,|_{x=3} = 1;$$

hence the curve cuts the axis of x between $x = 2$ and $x = 3$.

$$y\,|_{x=-1} = -3;$$

hence the curve cuts the axis between $x = 0$ and $x = -1$.

Incidentally we have shown that the cubic equation

$$x^3 - 3x^2 + 1 = 0$$

has three real roots, and we have located each between two successive integers.

EXERCISES

Discuss in a similar manner the following curves. In particular:

a) Determine the points at which the tangent is horizontal, if such exist, and draw the tangent at each of these points;

b) Determine the intervals in which the curve is concave upward, and those in which it is concave downward;

c) Determine the points of inflection, if any exist, and draw the tangent in each of these points;

d) Draw in the curve.*

In most cases it is desirable to take 2 cm. as the unit.

1. $$y = x^3 + 3x^2 - 2.$$

2. $$y = x^3 - 3x + 1.$$

3. $$y = x^3 + 3x + 1.$$

4. $$6y = 2x^3 - 3x^2 - 12x + 6.$$

5. $$6y = 2x^3 + 3x^2 - 12x - 4.$$

6. $$y = x^3 + x^2 + x + 1.$$

Suggestion. Show that the derivative has no real roots and hence, being continuous, never changes sign.

7. $$12y = 4x^3 - 6x^2 + 12x - 9.$$

8. $$y = 2x^2 - x - x^3.$$

9. $$12y = 4x^3 + 18x^2 + 27x + 12.$$

10. $$y = 1 - 4x + 6x^2 - 3x^3.$$

11. $$y = 1 + 2x + x^2 - x^3.$$

12. $4y = x^4 - 6x^2 + 8.$

13. $y = x^4 - 8x^2 + 4.$

14. $y = x - x^5.$

15. $y = x + x^5.$

16. $y = x^4 + x^2.$

17. $y = x^4 - x^2.$

18. $$y = 3x^5 + 5x^3 + 15x + 2.$$

19. $$60y = 2x^6 + 15x^4 + 60x^2 - 30.$$

6. Relative Maxima and Minima. Points of Inflection. A function

$$y = f(x) \tag{1}$$

* Since a curve separates very slowly from its tangent near a point of inflection, the material graph of the curve must necessarily coincide with the material graph of the tangent for some little distance.

is said to have a *maximum* at a point $x = x_0$ if its value at x_0 is larger than at any other point in the neighborhood of x_0. But such a maximum need not represent the largest value of the function in the complete interval $a \leqq x \leqq b$, as is shown by Fig. 27, and for this reason it is called a *relative maximum*, in distinction from a maximum maximorum, or an *absolute maximum*.

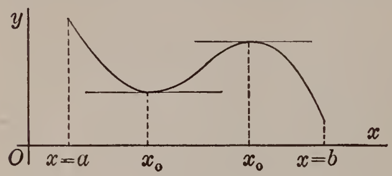

FIG. 27

A similar definition holds for a minimum, the word "larger" merely being replaced by "smaller."

It is obvious that a characteristic feature of a maximum is that the tangent there is parallel to the axis of x, the curve being concave downward. Similarly for a minimum, the curve here being concave upward. Hence the following

TEST FOR A MAXIMUM OR A MINIMUM. *If*

(a) $$[D_x y]_{x=x_0} = 0, \qquad [D_x^2 y]_{x=x_0} < 0,$$

the function has a maximum for $x = x_0$*; if*

(b) $$[D_x y]_{x=x_0} = 0, \qquad [D_x^2 y]_{x=x_0} > 0,$$

it has a minimum.

The condition is sufficient, but not necessary; cf. § 7.

Example. Let $$y = x^6 - 3x^2 + 1.$$

Here $$D_x y = 6x^5 - 6x = 6x(x^2 - 1)(x^2 + 1),$$

and hence $$D_x y = 0 \quad \text{for} \quad x = -1, 0, 1.$$

Thus the necessary condition for a maximum or a minimum, $D_x y = 0$, is satisfied at each of the points $x = -1, 0, 1$.

To complete the determination, if possible, compute the second derivative,

$$D_x^2 y = 30x^4 - 6,$$

and determine its sign at each of these points:

$$[D_x^2 y]_{x=-1} = 24 > 0, \qquad \therefore x = -1 \text{ gives a } \textit{minimum};$$
$$[D_x^2 y]_{x=0} = -6 < 0, \qquad \therefore x = 0 \text{ gives a } \textit{maximum};$$
$$[D_x^2 y]_{x=1} = 24 > 0, \qquad \therefore x = 1 \text{ gives a } \textit{minimum}.$$

Points of Inflection. A point of inflection is characterized geometrically by the phenomenon that, as a point P describes the curve, the tangent at P ceases rotating in the one direction and, turning back, begins to rotate in the opposite direction. Hence the slope of the curve, $\tan \tau$, has either a maximum or a minimum at a point of inflection.

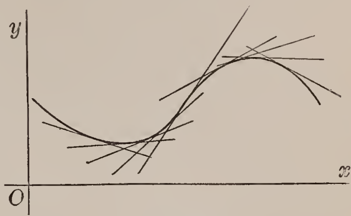

Fig. 28

Conversely, if $\tan \tau$ has a maximum or a minimum, the curve will have a point of inflection. For, suppose $\tan \tau$ is at a maximum when $x = x_0$. Then as x, starting with the value x_0, increases, $\tan \tau$, *i.e.* the slope of the curve, decreases algebraically, and so the curve is concave downward to the right of x_0. On the other hand, as x decreases, $\tan \tau$ also decreases, and so the curve is concave upward to the left of x_0.

Now, we have just obtained a theorem which insures us a maximum or a minimum in the case of any function which satisfies the conditions of the theorem. If, then, we choose as that function, $\tan \tau$, the theorem tells us that $\tan \tau$ will surely be at a maximum or a minimum if

$$D_x \tan \tau = 0, \qquad D_x^2 \tan \tau \neq 0.$$

Hence, remembering that

$$\tan \tau = D_x y,$$

we obtain the following

Test for a Point of Inflection. *If*

$$[D_x^2 y]_{x=x_0} = 0, \qquad [D_x^3 y]_{x=x_0} \neq 0,$$

the curve has a point of inflection at $x = x_0$.

This test, like the foregoing for a maximum or a minimum, is sufficient, but not necessary; cf. § 7.

Example. Let

$$27\,y = x^4 + 2\,x^3 - 12\,x^2 + 14\,x - 1.$$

Then

$$27\,D_x y = 4\,x^3 + 6\,x^2 - 24\,x + 14,$$

$$27\,D_x^2 y = 12\,x^2 + 12\,x - 24 = 12(x-1)(x+2),$$

$$27\,D_x^3 y = 12(2\,x+1).$$

Setting $D_x^2 y = 0$, we get the points $x = 1$ and $x = -2$. And since

$$27[D_x^3 y]_{x=1} = 36 \neq 0, \qquad 27[D_x^3 y]_{x=-2} = -36 \neq 0,$$

we see that both of these points are points of inflection.

The slope of the curve in these points is given by the equations:

$$27[D_x y]_{x=1} = 0, \qquad 27[D_x y]_{x=-2} = 54.$$

Hence the curve is parallel to the axis of x at the first of these points; at the second its slope is 2.

EXERCISES

Test the following curves for maxima, minima, and points of inflection, and determine the slope of the curve in each point of inflection.

1. $y = 4\,x^3 - 15\,x^2 + 12\,x + 1.$
2. $y = x^3 + x^4 + x^5.$
3. $6\,y = x^6 - 3\,x^4 + 3\,x^2 - 1.$
4. $y = (x-1)^3(x+2)^2.$

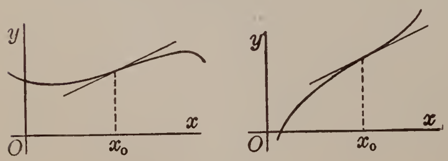

FIG. 29

5. $y = \dfrac{x}{2 + 3\,x^2}.$
6. $y = (1 - x^2)^3.$
7. Deduce a test for distinguishing between two such points of inflection as those indicated in Fig. 29.

7. Necessary and Sufficient Conditions. In order to understand the nature of the tests obtained in the foregoing paragraph it is essential that the student have clearly in mind the meaning of a *necessary condition* and of a *sufficient condition.* Let us illustrate these ideas by means of some simple examples.

a) A *necessary* condition that a quadrilateral be a square is that its angles be right angles. But the condition is obviously not sufficient; all rectangles also satisfy it.

b) A *sufficient* condition that a quadrilateral be a square is that its angles be right angles and each side be 4 in. long. But the condition is obviously not necessary; the sides might be 6 in. long.

c) A *necessary and sufficient* condition that a quadrilateral be a square is that its angles be right angles and its sides be mutually equal.

As a further illustration consider the following. It is a well-known fact about whole numbers that if the sum of the digits of a whole number is divisible by 3, the number is divisible by 3; and conversely. Also, if the sum of the digits of a whole number is divisible by 9, the number is divisible by 9; and conversely. Hence we can say:

i) A *necessary* condition that a whole number be divisible by 9 is that the sum of its digits be divisible by 3. But the condition is not sufficient.

ii) A *sufficient* condition that a whole number be divisible by 3 is that the sum of its digits be divisible by 9. But the condition is not necessary.

iii) A *necessary and sufficient* condition that a whole number be divisible by 3 (or 9) is that the sum of its digits be divisible by 3 (or 9).

Turning now to the considerations of § 6, we see that a *necessary* condition for a minimum is that

$$D_x y = 0$$

at the point in question, $x = x_0$. But this condition is not sufficient. When it is fulfilled, the function may have a maximum, or it may have a point of inflection with horizontal tangent.

On the other hand, the condition

$$[D_x y]_{x=x_0} = 0, \qquad [D_x^2 y]_{x=x_0} > 0$$

is *sufficient* for a minimum. But it is not necessary. Thus the function

$$(1) \qquad y = x^4$$

obviously has a minimum when $x = 0$. The necessary condition, $D_x y = 0$, is of course fulfilled:

$$D_x y = 4\,x^3, \qquad [D_x y]_{x=0} = 0.$$

But here $\quad D_x^2 y = 12\,x^2, \quad$ and $\quad [D_x^2 y]_{x=0}$

is not positive; it is 0.

Again, as was shown in § 4, a *necessary* condition for a point of inflection is that

$$D_x^2 y = 0$$

at that point. But this condition is not sufficient. Thus in the case of the curve (1) this condition is fulfilled at the origin. But the origin is not a point of inflection.

Remark. It may seem to the student that such tests are unsatisfactory since they do not apply to all cases and thus appear to be incomplete. But their very strength lies in the fact that they do not tell the truth in too much detail. They single out the big thing in the cases which arise in practice and yield criteria which can be applied with ease to the great majority of these cases.

8. Velocity; Rates. By the *average velocity* with which a point moves for a given length of time t is meant the distance s traversed divided by the time:

$$\text{average velocity} = \frac{s}{t}.$$

Thus a railroad train which covers the distance between two stations 15 miles apart in half an hour has an average speed of $15/\frac{1}{2} = 30$ miles an hour.

When, however, the point in question is moving sometimes fast and sometimes slowly, we can describe its speed approximately at any given instant by considering a short interval of time immediately succeeding the instant t_0 in question, and taking the average velocity for this short interval.

For example, a stone dropped from rest falls according to the law:

$$s = 16t^2.$$

To find how fast it is going after the lapse of t_0 seconds. Here

(1) $$s_0 = 16t_0^2.$$

A little later, at the end of t' seconds from the beginning of the fall,

(2) $$s' = 16t'^2$$

and the average velocity for the interval of $t' - t_0$ seconds is

(3) $$\frac{s' - s_0}{t' - t_0} \text{ ft. per second.}$$

Let us consider this average velocity, in particular, after the lapse of 1 second:

$$t_0 = 1, \qquad s_0 = 16.$$

Let the interval of time, $t' - t_0$, be $\frac{1}{10}$ sec. Then

$$s' = 16 \times 1.1^2 = 19.36,$$

$$\frac{s' - s_0}{t' - t_0} = \frac{3.36}{.1} = 33.6 \text{ ft. a second.}$$

Thus the average velocity for one-tenth of a second immediately succeeding the end of the first second of fall is 33.6 ft. a second.

Next, let the interval of time be $\frac{1}{100}$ sec. Then a similar computation gives, to three significant figures:

$$\frac{s' - s_0}{t' - t_0} = 32.2 \text{ ft. a second.}$$

And when the interval is taken as $\frac{1}{1000}$ sec., the average velocity is 32.0 ft. a second.

These numerical results indicate that we can get at the speed of the stone at any desired instant to any desired degree of accuracy by direct computation; we need only to reckon out the average velocity for a sufficiently short interval of time succeeding the instant in question.

We can proceed in a similar manner when a point moves according to any given law. Can we not, however, by the aid of the Calculus avoid the labor of the computations and at the same time make precise exactly what is meant by *the velocity* of the point *at a given instant*? If we regard the interval of time $t' - t_0$ as an increment of the variable t and write $t' - t_0 = \Delta t$, then $s' - s_0 = \Delta s$ will represent the corresponding increment in the function, and thus we have:

$$\text{average velocity} = \frac{\Delta s}{\Delta t}.$$

Now allow Δt to approach 0 as its limit. Then the average velocity will in general approach a limit, and *this limit we take as the definition of the velocity*, v, *at the instant* t_0:

$$\lim \text{ (average velocity from } t = t_0 \text{ to } t = t')$$

$$= \text{actual velocity * at instant } t = t_0,$$

or

$$v = \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} = D_t s.$$

Hence it appears that the velocity of a point is the *time-derivative* of the space it has traveled. In the case of a freely falling body this velocity is

$$v = D_t s = 32\, t.$$

In the foregoing definition, s has been taken as the distance actually traversed by the moving point, P. More generally, let s denote the length of the arc of the curve on which P is moving, s being measured from an arbitrarily chosen fixed

* Sometimes called the *instantaneous velocity*.

point of that curve. Either direction along the curve may be chosen as the positive sense for s. Thus, in the case of a freely falling body, s might be taken as the distance of the body above the ground. If h denotes the initial distance, then

$$s + s' = h,$$

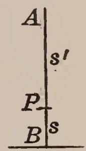

FIG. 30

where s' denotes the distance actually traversed by P at any given instant. Hence

$$D_t s + D_t s' = 0,$$

or

$$D_t s = - D_t s'.$$

Here $D_t s$ gives *numerically* the value of the velocity, but $D_t s$ is a negative quantity.

We will, accordingly, extend the conception of velocity, defining the velocity v of the point as $D_t s$:

$$v = D_t s.$$

Thus the numerical value of v or $D_t s$ will always give the *speed*, or the value of the velocity in the earlier sense. In case s increases with the time, $D_t s$ is positive and represents the speed. If, however, s decreases with the time, $D_t s$ is negative, and the velocity, v, is therefore here negative, the speed now being given by $-v$ or $-D_t s$. In all cases,

$$\text{Speed} = |v| = |D_t s|.$$

Example. Let a body be projected upward with an initial velocity of 96 ft. a second. Assuming from Physics the law that

$$s = 96t - 16t^2,$$

find its velocity $a)$ at the end of 2 sec.

$b)$ at the end of 5 sec.

Solution. By definition, the velocity at any instant is

$$v = D_t s = 96 - 32t.$$

Hence

$a)$ $$v\,|_{t=2} = 64.$$

$b)$ $$v\,|_{t=5} = -64.$$

The meaning of these results is that, at each of the two instants, the speed is the same, namely, 64 ft. a second (and the height above the ground is also seen to be the same, $s = 128$ ft.). But when $t = 2$, $D_t s$ is positive; hence s is increasing with the time and the body is rising. When $t = 5$, $D_t s$ is negative; hence s is decreasing with the time, and the body is descending.

Rates. Consider any length or distance, r, which is changing with the time, and so is a function of the time. Let r_0 denote the value of r at a given instant, $t = t_0$, and let r' be the value of r at a later instant, $t = t'$. Then the increase in r will be $r' - r_0 = \Delta r$ and that in t will be $t' - t_0 = \Delta t$. Thus in the interval of time of Δt seconds succeeding the instant $t = t_0$,

$$\text{average rate of increase of } r = \frac{\Delta r}{\Delta t}.$$

Now let Δt approach 0 as its limit. Then the average rate of increase will in general approach a limit, and *this limit we take as the definition of the rate of increase of r at the instant* t_0:

$$\lim\,(\text{average rate of increase from } t = t_0 \text{ to } t = t')$$

$$= \text{actual rate of increase at instant } t = t_0$$

$$= \lim_{\Delta t \doteq 0} \frac{\Delta r}{\Delta t} = D_t r.$$

In other words, the *rate at which r is increasing* at any instant is defined as the *time-derivative* of r.

If r is decreasing, $D_t r$ will be a negative quantity; and conversely, if $D_t r$ is negative, then r is decreasing. In either case, the *numerical* value of $D_t r$ gives the *rate of change* of r; just as, in the case of velocities, the numerical value of $D_t s$ gives the speed.

More generally, instead of r, we may have any physical quantity, u, as an area or a volume or the current in an electric circuit or the number of calories in a given body.

In all these cases, the rate at which u is increasing is defined as the *time-derivative* of u, *i.e.* as $D_t u$; and the rate of change of u is $|D_t u|$.

Example. At noon, one ship is steaming east at the rate of 18 miles an hour, and a second ship, 40 miles north of the first, is steaming south at the rate of 20 miles an hour. At what rate are they separating from each other at one o'clock?

Solution. The relation between r and t is here given by the Pythagorean Theorem:

$$r^2 = (40 - 20t)^2 + (18t)^2,$$

or

(1) $$r^2 = 1600 - 1600t + 724t^2.$$

Hence

(2) $$r = \sqrt{1600 - 1600t + 724t^2}.$$

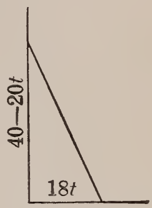

FIG. 31

We wish to find $D_t r$. This can be done by differentiating equation (2); but that would be poor technique, since it is simpler to differentiate equation (1) through with respect to t:

$$2rD_t r = -1600 + 1448t,$$

(3) $$D_t r = \frac{-800 + 724t}{r}.$$

Equation (3) gives the rate at which r is increasing at any instant t; *i.e.* t hours past noon, or at t o'clock.

Setting now, in particular, $t = 1$, we obtain:

$$D_t r\,|_{t=1} = -\frac{76}{\sqrt{724}} = -2.825.$$

The meaning of this result is twofold. First, since $D_t r$ is negative when $t = 1$, the ships are not receding from each other, but are coming nearer together. Secondly, the rate of change of the distance between them is, at one o'clock, 2.825 miles an hour.

Let the student determine how long they will continue to approach each other, and what the shortest distance between them will be.

Remark. It is important for the student to reflect on the method of solution of this problem, since it is typical. We were asked to find the rate of recession *at just one instant*, $t=1$. We began by determining the rate of recession generally, *i.e.* for an *arbitrary instant*, $t=t$. Having solved the general problem, we then, as the last step in the process, brought into play the specific value of t which alone we cared for, namely, $t=1$.

The student will meet this method again and again, — in integration, in mechanics, in series, etc. We can formulate the foregoing remark suggestively as follows: By means of the Calculus we can often determine a particular physical quantity, like a velocity, an area, or the time it takes a body, acted on by known forces, to reach a certain position. The method consists in first determining a *function*, whereby the general problem is solved for the variable case; and then, as the last step in the process, the special numerical values with which alone the proposed question is concerned, are brought into play.

EXERCISES

1. The height of a stone thrown vertically upward is given by the formula:

$$s = 48t - 16t^2.$$

When it has been rising for one second, find (*a*) its average velocity for the next $\frac{1}{10}$ sec.; (*b*) for the next $\frac{1}{100}$ sec.; (*c*) its actual velocity at the end of the first second; (*d*) how high it will rise.

Ans. (*a*) 14.4 ft. a second; (*b*) 15.84 ft. a second; (*c*) 16 ft. a second; (*d*) 36 ft.

2. One ship is 80 miles due south of another ship at noon, and is sailing north at the rate of 10 miles an hour. The

second ship sails west at the rate of 12 miles an hour. Will the ships be approaching each other or receding from each other at 2 o'clock? What will be the rate at which the distance between them is changing at that time? How long will they continue to approach each other?

3. If two ships start abreast half a mile apart and sail due north at the rates of 9 miles an hour and 12 miles an hour, how far apart will they be at the end of half an hour? How fast will they be receding at that time?

4. Two ships are steaming east, one at the rate of 18 miles an hour, the other at the rate of 24 miles an hour. At noon, one is 50 miles south of the other. How fast are they separating at 7 P.M.?

5. A ladder 20 ft. long rests against a house. A man takes hold of the lower end of the ladder and walks off with it at the uniform rate of 2 ft. a second. How fast is the upper end of the ladder coming down the wall when the man is 4 ft. from the house?

6. A kite is 150 ft. high and there are 250 ft. of cord out. If the kite moves horizontally at the rate of 4 m. an hour directly away from the person who is flying it, how fast is the cord being paid out? *Ans.* $3\frac{1}{5}$ m. an hour.

7. A stone is dropped into a placid pond and sends out a series of concentric circular ripples. If the radius of the outer ripple increases steadily at the rate of 6 ft. a second, how rapidly is the area of the water disturbed increasing at the end of 2 sec.? *Ans.* 452 sq. ft. a second.

8. A spherical raindrop is gathering moisture at such a rate that the radius is steadily increasing at the rate of 1 mm. a minute. How fast is the volume of the drop increasing when the diameter is 2 mm.?

9. A man is walking over a bridge at the rate of 4 miles an hour, and a boat passes under the bridge immediately below him rowing 8 miles an hour. The bridge is 20 ft. above the

boat. How fast are the boat and the man separating 3 minutes later?

Suggestion. The student should make a space model for this problem by means, for example, of the edge of a table, a crack in the floor, and a string; or by two edges of the room which do not intersect, and a string. He should then make a drawing of his model such as is here indicated.

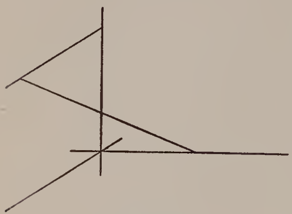

Fig. 32

10. A locomotive running 30 miles an hour over a high bridge dislodges a stone lying near the track. The stone begins to fall just as the locomotive passes the point where it lay. How fast are the stone and the locomotive separating 2 sec. later? *

11. Solve the same problem if the stone drops from a point 40 ft. from the track and at the same level, when the locomotive passes.

12. A lamp-post is distant 10 ft. from a street crossing and 60 ft. from the houses on the opposite side of the street. A man crosses the street, walking on the crossing at the rate of 4 miles an hour. How fast is his shadow moving along the walls of the houses when he is halfway over?

* Bôcher, *Plane Analytic Geometry*, p. 230.

CHAPTER IV

INFINITESIMALS AND DIFFERENTIALS

1. Infinitesimals. An *infinitesimal* is a variable which it is desirable to consider only for values numerically small and which, when the formulation of the problem in hand has progressed to a certain stage, is allowed to approach 0 as its limit.

Thus in the problem of differentiation, or finding the limit

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = D_x y, \tag{1}$$

Δx and Δy are infinitesimals; for we allow Δx to approach 0 as its limit, and then Δy also approaches 0..

Again, if we denote the value of the difference $\Delta y/\Delta x - D_x y$ by ϵ, so that

$$\frac{\Delta y}{\Delta x} - D_x y = \epsilon, \tag{2}$$

then ϵ is an infinitesimal. For, when Δx approaches 0, the left-hand side of equation (2) approaches 0, and so ϵ is a variable which approaches 0 as its limit, *i.e.* an infinitesimal.

Principal Infinitesimal. When we are dealing with a number of infinitesimals, α, β, γ, etc., it is usually possible to choose any one of them as the independent variable, the others then becoming functions of it, or dependent variables. That infinitesimal which is chosen as the independent variable is called the *principal infinitesimal.*

Thus, if the infinitesimals are α and β, and if

$$\beta = \frac{2\alpha}{1 + 3\alpha}, \tag{3}$$

it is natural to choose α as the principal infinitesimal. But it is perfectly possible to take β as the principal infinitesimal. α then becomes the dependent variable, and is expressed in terms of β by solving equation (3) for α:

$$\alpha = \frac{\beta}{2 - 3\beta}. \tag{4}$$

Order of Infinitesimals. We are going to separate infinitesimals into classes, according to the relative speed with which they approach 0. Suppose we let α set the pace, taking on the values .5, .1, .01, .001, etc. Consider, for example, α^2. Then α^2 takes on the respective values .25, .01, .0001, etc., and hence runs far ahead of α:

α	.5	.1	.01	.001 ...
α^2	.25	.01	.0001	.000001 ...

Furthermore, the closer the two get to 0, the *relatively* nearer α^2 is to 0. Thus, when $\alpha = .5$, α^2 is twice as close; but when $\alpha = .01$, α^2 is one hundred times as close; and so on.

Again, consider the infinitesimal $\frac{1}{2}\alpha$. It is always twice as close to 0 as α is. Similarly, 10α is always one-tenth as close as α.

From these examples we see that there is a decided difference between the *relative* behavior of α and $k\alpha$ on the one hand, and that of α and α^2 on the other. For, $k\alpha$ is keeping pace *relatively* with α, whereas α^2 runs indefinitely ahead of α, *relatively*. Consequently, we should put $k\alpha$ into the same class with α, whereas α^2 forms the starting point for a new class. To this latter class would belong such infinitesimals as $\frac{1}{2}\alpha^2$ or $4\alpha^2 - \alpha^3$; and the former class would include, for example, $2\alpha + 3\alpha^2$ and $\frac{-1}{100}\alpha - 1000\alpha^8$. Let the student make out a table like the above for each of these examples.

What is the common property of all infinitesimals of the same class? Is it not, that, for two infinitesimals, the *relative* speed with which they approach 0 is nearly, or quite, a fixed number not zero? It is this idea which lies at the bottom of

the conception of the *order* of an infinitesimal, and it is formulated in a precise definition as follows:

DEFINITION. Two infinitesimals, β and γ, are said to be of the *same order* if their ratio approaches a limit not 0:

$$\lim \frac{\beta}{\gamma} = K \neq 0.$$

Thus $\qquad \beta = 2\alpha + \alpha^2 \qquad$ and $\qquad \gamma = 3\alpha - \alpha^3$

are of the same order. For,

$$\frac{\beta}{\gamma} = \frac{2\alpha + \alpha^2}{3\alpha - \alpha^3} = \frac{2 + \alpha}{3 - \alpha^2},$$

and hence, when α approaches 0,

$$\lim \frac{\beta}{\gamma} = \lim \frac{2 + \alpha}{3 - \alpha^2} = \frac{2}{3} \neq 0.$$

Similarly, $12\alpha^2 + 3\alpha^5$ and $6\alpha^2 - 7\alpha^3$ are infinitesimals of the same order.

An infinitesimal β is said to be of *higher order* than γ if

$$\lim \frac{\beta}{\gamma} = 0,$$

Thus if $\qquad \beta = 9\alpha^2 \qquad$ and $\qquad \gamma = 2\alpha + 5\alpha^4,$

β is of higher order than α. For,

$$\frac{\beta}{\gamma} = \frac{9\alpha^2}{2\alpha + 5\alpha^4} = \frac{9\alpha}{2 + 5\alpha^3},$$

and hence, when α approaches 0,

$$\lim \frac{\beta}{\gamma} = \lim \frac{9\alpha}{2 + 5\alpha^3} = 0.$$

Finally, β is said to be of *lower order* than γ if

(5) $$\lim \frac{\beta}{\gamma} = \infty,$$

(read: "*β/γ becomes infinite*"; NOT "β/γ equals infinity." *).

* The student should now turn back to Chapter II, § 5, and read again carefully what is said there about infinity. In particular, he should im-

Thus if $\beta = \sqrt{\alpha}$ and $\gamma = 6\alpha + \alpha^3$,

β is of lower order than γ. For

$$\frac{\beta}{\gamma} = \frac{\sqrt{\alpha}}{6\alpha + \alpha^3} = \frac{1}{\sqrt{\alpha}(6 + \alpha^3)}.$$

When α approaches 0, it is evident that the last fraction increases without limit, or

$$\lim \frac{\beta}{\gamma} = \infty.$$

First Order, Second Order, etc. An infinitesimal β is said to be of the *first order* if it is of the same order as the principal infinitesimal, α; *i.e.* if

$$\lim \frac{\beta}{\alpha} = K \neq 0.$$

If β is of the same order as α^2, *i.e.* if

$$\lim \frac{\beta}{\alpha^2} = K \neq 0,$$

then β is said to be of the *second order*. And, generally, if β is of the same order as α^n, *i.e.* if

$$\lim \frac{\beta}{\alpha^n} = K \neq 0,$$

then β is said to be of the *n-th order*.

Thus if

$$\beta = 2\alpha \quad \text{or} \quad \beta = \frac{\alpha}{2 - \alpha} \quad \text{or} \quad \beta = \alpha + \alpha^2,$$

then β is of the first order.

But if

$$\beta = 2\alpha^2 + \alpha^3 \quad \text{or} \quad \beta = \frac{\alpha^2}{3 + \alpha} \quad \text{or} \quad \beta = \alpha^2,$$

then β is of the second order.

press on his mind the fact that infinity is not a limit and that in the notation used in (5) the = sign does *not* mean that one number is equal to another number. The formula is not an equation in the sense in which $2x = 3$ or $a^2 - b^2 = (a - b)(a + b)$ is an equation. The formula means no more and no less than that the variable β/γ increases in value without limit.

If $\beta = \sqrt{a}$, then
$$\frac{\beta}{\alpha^{\frac{1}{2}}} = 1,$$
and
$$\lim \frac{\beta}{\alpha^{\frac{1}{2}}} = 1 \neq 0.$$
Hence β is of the order $\frac{1}{2}$.

It is easily seen that if two infinitesimals β and γ are, under the present definition, each of order n, then they also satisfy the earlier definition of being of the same order. For, let

$$\lim \frac{\beta}{\alpha^n} = K \neq 0 \qquad \text{and} \qquad \lim \frac{\gamma}{\alpha^n} = L \neq 0.$$

Then, if we denote the differences $\beta/\alpha^n - K$ and $\gamma/\alpha^n - L$ respectively by ϵ and η, so that

(6) $$\frac{\beta}{\alpha^n} - K = \epsilon \qquad \text{and} \qquad \frac{\gamma}{\alpha^n} - L = \eta,$$

these variables, ϵ and η, will be infinitesimals. For, the left-hand side of each of the equations (6) approaches 0.

From equations (6) it follows that

$$\frac{\beta}{\alpha^n} = K + \epsilon \qquad \text{and} \qquad \frac{\gamma}{\alpha^n} = L + \eta.$$

On dividing one of these equations by the other we have:

$$\frac{\beta}{\gamma} = \frac{K + \epsilon}{L + \eta}.$$

We are now ready to allow α to approach 0 as its limit. Then

$$\lim \frac{\beta}{\gamma} = \lim \frac{K + \epsilon}{L + \eta}.$$

By Theorem III of Chapter 2, § 5 this last limit has the value

$$\lim \frac{K + \epsilon}{L + \eta} = \frac{\lim (K + \epsilon)}{\lim (L + \eta)} = \frac{K}{L}.$$

Hence, finally
$$\lim \frac{\beta}{\gamma} = \frac{K}{L} \neq 0,$$
q. e. d.

EXERCISES

1. Show that

$$\beta = 5\alpha - 11\alpha^2 + \alpha^3 \qquad \text{and} \qquad \gamma = 7\alpha + \alpha^4$$

are infinitesimals of the same order.

2. Show that

$$\beta = 2\alpha - 3\alpha^2 \qquad \text{and} \qquad \gamma = 2\alpha + \alpha^4$$

are infinitesimals of the same order, but that their difference, $\beta - \gamma$, is of higher order than β (or γ).

3. Show that $\beta = \dfrac{7\alpha^2}{\alpha^3 - 2}$ is an infinitesimal of the second order, referred to α as principal infinitesimal.

4. Show that $\beta = \sqrt{\alpha^2 + 2\alpha^5}$ is of the first order, referred to α.

5. Show that $\beta = \sqrt{2\alpha + 13\alpha^3}$ is of lower order than α.

6. Show that the order of β in question 5 is $n = \frac{1}{2}$.

Determine the order of each of the following infinitesimals, referred to α as the principal infinitesimal:

7. $\frac{1}{2}\alpha + 18\alpha^3$.

8. $-\alpha + \sqrt{2\alpha^3 + \alpha^4}$.

9. $\dfrac{7\alpha^2}{13 - \alpha}$.

10. $\sqrt{\dfrac{2\alpha^2 + \alpha^5}{8 - 7\alpha}}$.

11. $\sqrt[3]{\alpha^3 - \alpha}$.

12. $\sqrt[13]{-\alpha^{12} + \alpha^{13}}$.

13. $\sqrt[5]{2\alpha^2 - \alpha^3}$.

14. $\sqrt[4]{\dfrac{3\alpha^6 + 4\alpha^4}{\alpha^2 + 2}}$.

15. If β and γ are infinitesimals of orders n and m respectively, show that their product, $\beta\gamma$, is an infinitesimal of order $n + m$.

16. If β and γ are infinitesimals of the same order, show that their sum is, in general, an infinitesimal of the same order.

Are there exceptions? Illustrate by examples.

2. Continuation; Fundamental Theorem. *Principal Part of an Infinitesimal.* Let β be an infinitesimal of order n, and let α be the principal infinitesimal. Then

$$\lim \frac{\beta}{\alpha^n} = K \neq 0.$$

Moreover, as pointed out in the last paragraph,

$$\frac{\beta}{\alpha^n} = K + \epsilon, \tag{1}$$

where ϵ is infinitesimal. From (1) it follows that

$$\beta = K\alpha^n + \epsilon\alpha^n. \tag{2}$$

This last equation gives a most important analysis (*i.e.* breaking up) of β into two parts, each of which is simple for its own peculiar reason.

i) $K\alpha^n$ is the simplest infinitesimal of the nth order imaginable, — a monomial in the independent variable, the function

$$y = Kx^n.$$

ii) $\epsilon\alpha^n$ is an infinitesimal of higher order than the nth.

The first part, $K\alpha^n$, is called the *principal part* of β.

By far the most important case in practice is that of infinitesimals β of the *first* order, $n = 1$. Here

$$\frac{\beta}{\alpha^n} = \frac{\beta}{\alpha} = K + \epsilon$$

and

$$\beta = K\alpha + \epsilon\alpha.$$

Hence we see that *the principal part of an infinitesimal of the first order is proportional to the principal infinitesimal.*

Example 1. Let $\beta = 2\alpha - \alpha^2.$

Then β is obviously of the first order, or $n = 1$, and here

$$\frac{\beta}{\alpha^n} = \frac{\beta}{\alpha} = 2 - \alpha,$$

Clearly, then, $K = 2, \qquad \epsilon = -\alpha,$

and the principal part of β is 2α.

Example 2. Let
$$\beta = \frac{2\alpha^2}{7 - 4\alpha}.$$

Here, obviously, $n = 2$, and
$$\lim \frac{\beta}{\alpha^2} = \lim \frac{2}{7 - 4\alpha} = \frac{2}{7}.$$

Hence $K = \frac{2}{7}$. By definition,
$$\epsilon = \frac{\beta}{\alpha^n} - K.$$

In the present case, then,
$$\epsilon = \frac{2}{7 - 4\alpha} - \frac{2}{7} = \frac{8\alpha}{7(7 - 4\alpha)}.$$

EXERCISE

Determine the principal parts of a goodly number of the infinitesimals occurring in the Exercises at the end of § 1.

Equivalent Infinitesimals. Two infinitesimals, as β and γ, shall be said to be *equivalent* if the limit of their ratio is unity:
$$\lim \frac{\beta}{\gamma} = 1.$$

For example, the following pairs of infinitesimals are equivalent:

i) $2\alpha + \alpha^2$ and $2\alpha + \alpha^3$;

ii) $\frac{1}{2}\alpha^2 - \alpha^3$ and $\frac{1}{2}\alpha^2 + \alpha^3$;

iii) $\sqrt{2\alpha + 5\alpha^2}$ and $\sqrt{2\alpha - 7\alpha^4}$.

An infinitesimal and its principal part are always equivalent infinitesimals. For, if $K\alpha^n$ is the principal part of β, then
$$\beta = K\alpha^n + \eta,$$

where η is of higher order than $K\alpha^n$. Hence

$$\frac{\beta}{K\alpha^n}=1+\frac{\eta}{K\alpha^n}, \qquad \lim\frac{\beta}{K\alpha^n}=1+\lim\frac{\eta}{K\alpha^n}.$$

But $\lim \eta/K\alpha^n = 0$, and the statement is established.

Two infinitesimals which have the same principal parts are equivalent, and conversely.

Equivalent infinitesimals are of the same order; but the converse is not true.

The difference between two equivalent infinitesimals, β and γ, namely, $\beta-\gamma$, is of higher order than β or γ. For

$$\frac{\beta-\gamma}{\gamma}=\frac{\beta}{\gamma}-1;$$

hence

$$\lim\frac{\beta-\gamma}{\gamma}=\lim\left(\frac{\beta}{\gamma}-1\right)$$

$$=\left(\lim\frac{\beta}{\gamma}\right)-1=0, \qquad \text{q. e. d.}$$

Conversely, if β and γ are two infinitesimals whose difference, $\beta-\gamma$, is of higher order than β or γ, then β and γ are equivalent.

For, since

$$\frac{\beta-\gamma}{\gamma}=\frac{\beta}{\gamma}-1,$$

it follows that

$$\lim\left(\frac{\beta}{\gamma}-1\right)=\lim\frac{\beta-\gamma}{\gamma}.$$

The right-hand side of this equation is 0 by hypothesis, and the left-hand side is equal to

$$\left(\lim\frac{\beta}{\gamma}\right)-1.$$

Hence

$$\lim\frac{\beta}{\gamma}=1, \qquad \text{q. e. d.}$$

We come now to a theorem of prime importance in the Infinitesimal Calculus.

Fundamental Theorem. *The limit of the ratio of two infinitesimals,*

$$\lim \frac{\beta}{\gamma},$$

is unchanged if the numerator infinitesimal β be replaced by any equivalent infinitesimal β' and the denominator infinitesimal γ be replaced by any equivalent infinitesimal γ'.

In other words:

$$\lim \frac{\beta}{\gamma} = \lim \frac{\beta'}{\gamma'}$$

provided $\quad \lim \frac{\beta}{\beta'} = 1 \quad$ *and* $\quad \lim \frac{\gamma}{\gamma'} = 1.$

The proof is immediate. It is obvious that

$$\frac{\beta'}{\gamma'} = \frac{\beta'\beta\gamma}{\beta\gamma\gamma'}.$$

Hence by Theorem II, Chapter II, § 5 we have

$$\lim \frac{\beta'}{\gamma'} = \left(\lim \frac{\beta'}{\beta}\right)\left(\lim \frac{\beta}{\gamma}\right)\left(\lim \frac{\gamma}{\gamma'}\right).$$

But the first and third limits on the right-hand side are each equal to 1 by hypothesis. Hence

$$\lim \frac{\beta'}{\gamma'} = \lim \frac{\beta}{\gamma}, \qquad \text{q. e. d.}$$

The theorem can be stated in the following equivalent form:

The limit of the ratio of two infinitesimals is the same as the limit of the ratio of their principal parts.

The student must not generalize from this theorem and infer that an infinitesimal can always and for all purposes be replaced by an equivalent infinitesimal. Thus if

$$\beta = 2\alpha + \alpha^3 \qquad \text{and} \qquad \gamma = 2\alpha - \alpha^2,$$

their difference, $\quad \beta - \gamma = \alpha^3 + \alpha^2,$

is an infinitesimal of the second order. On the other hand,

$$\gamma' = 2\alpha$$

is equivalent to γ. But it is not true that the difference of β and γ', namely,

$$\beta - \gamma' = \alpha^3,$$

is an infinitesimal of the second order. It is obviously of order 3. Thus replacing γ by an equivalent infinitesimal has here changed the order of the difference $\beta - \gamma$.

3. Differentials. Let $\quad y = f(x)$

be a function of x, and let $D_x y$ be its derivative:

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = D_x y.$$

Let the difference $\Delta y / \Delta x - D_x y$ be denoted by ϵ. Then

$$\frac{\Delta y}{\Delta x} = D_x y + \epsilon,$$

and

$$(1) \qquad \Delta y = D_x y \Delta x + \epsilon \Delta x.$$

Since x is the independent variable, Δx can be taken as the principal infinitesimal. $D_x y$ does not vary with Δx; it is a constant, for we are considering its value at a fixed point $x = x_0$. Since, moreover, $D_x y$ is not in general zero, equation (1) represents Δy as the sum of its principal part, $D_x y \Delta x$, and an infinitesimal of higher order, $\epsilon \Delta x$.

Definition of a Differential. The expression $D_x y \Delta x$ is called the *differential* of the function, and is denoted by dy:

$$(2) \qquad dy = D_x y \Delta x, \qquad \text{or} \qquad df(x) = D_x f(x) \Delta x.$$

(read: "differential y" or "differential $f(x)$" or "dy," etc.).

Thus if $\qquad y = x^2,$

$$dy = 2x\Delta x, \qquad \text{or} \qquad dx^2 = 2x\Delta x.$$

Since the definition (2) holds for every function $y = f(x)$, it can be applied to the particular function

$$f(x) = x.$$

Hence

(3) $$dx = D_x x \Delta x = \Delta x.$$

But it is not in general true that Δy and dy are equal, since ϵ is in general different from 0. Thus we see that *the differential of the independent variable is equal to the increment of that variable; but the differential of the dependent variable is not in general equal to the increment of that variable.*

By means of (3) equation (2) can now be written in the form

(4) $$dy = D_x y \, dx.$$

Hence

(5) $$\frac{dy}{dx} = D_x y.$$

Geometrically, the increment Δy of the function is represented by the line MP', Fig. 33; and the differential, dy, is equal to MQ, for from (5)

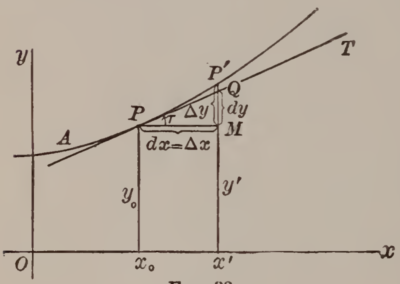

FIG. 33

$$\tan \tau = \frac{dy}{dx}$$

or $$dy = dx \tan \tau.$$

In other words, Δy represents the distance from the level of P to the *curve*, when $x = x'$; dy, the distance from the level of P to the *tangent*. Moreover, the difference

$$\Delta y - dy = \epsilon \Delta x$$

is shown geometrically as the line QP', and is obviously from the figure an infinitesimal of higher order than $\Delta x = PM$.

It is also clear from the figure that Δy and dy are equal when and only when the curve $y = f(x)$ is a straight line; *i.e.*

when $f(x)$ is a linear function,

$$f(x) = ax + b.$$

Hitherto x has been taken as the independent variable, Δx as the principal infinitesimal. We come now to the theorem on which the whole value of differentials for the purpose of performing differentiation depends.

THEOREM. *The relation* (4):

$$dy = D_x y\, dx,$$

is true, even when x and y are both dependent on a third variable, t.

Suppose, namely, that x and y come to us as functions of a third variable, t:

(6) $$x = \phi(t), \qquad y = \psi(t),$$

and that, when we eliminate t between these two equations, we obtain the function

$$y = f(x).$$

Then dx and dy have the following values, in accordance with the above definition, since t, not x, is now the independent variable, Δt the principal infinitesimal:

$$dy = D_t y\, \Delta t, \qquad dx = D_t x\, \Delta t.$$

We wish to prove that

$$dy = D_x y\, dx.$$

Now by Theorem V of Chap. II, § 5:

$$D_t y = D_x y\, D_t x.$$

Hence, multiplying through by Δt, we get:

$$D_t y\, \Delta t = D_x y \cdot D_t x\, \Delta t,$$

or

$$dy = D_x y\, dx, \qquad \text{q. e. d.}$$

With this theorem the explicit use of Theorem V in Chap. II, § 5 disappears, Formula V of that theorem now taking on the form of an algebraic identity:

$$\frac{du}{dx}=\frac{du}{dy}\frac{dy}{dx}.$$

To this fact is due the chief advantage of differentials in the technique of differentiation.

Differentials of Higher Order. It is possible to introduce differentials of higher order by a similar definition:

$$(7)\qquad d^2y = D_x{}^2y\,\Delta x^2, \qquad d^3y = D_x{}^3y\,\Delta x^3, \qquad \text{etc.},$$

x being the independent variable. We should then have by (3)

$$(8)\qquad d^2y = D_x{}^2y\,dx^2 \qquad \text{or} \qquad \frac{d^2y}{dx^2} = D_x{}^2y, \qquad \text{etc.}$$

Unfortunately, however, relation (8) does not continue to hold when x and y both depend on a third variable, t. For example, suppose

$$x = t^2, \qquad y = a + t^2.$$

Then

$$y = a + x.$$

When t is taken as the independent variable, we have according to relation (8):

$$d^2y = D_t{}^2y\,dt^2 = 2\,dt^2;$$

and since

$$dx = 2\,t\,dt,$$

it follows that

$$\frac{d^2y}{dx^2}=\frac{2\,dt^2}{4\,t^2\,dt^2}=\frac{1}{2\,t^2}=\frac{1}{2\,x}.$$

On the other hand, when x is taken as the independent variable, relation (8) becomes

$$d^2y = D_x{}^2y\,dx^2 = 0,$$

and consequently

$$\frac{d^2y}{dx^2}=0.$$

Thus the quotient, $\dfrac{d^2y}{dx^2}$, is seen to have two entirely distinct values according as t or x is taken as the independent variable. We will agree, therefore, to discard this definition. The notation $\dfrac{d^2y}{dx^2}$ as meaning $D_x{}^2y$ is, however, universally used in the Calculus, and so we will accept the definitions

$$\frac{d^2y}{dx^2}=D_x{}^2y, \qquad \frac{d^3y}{dx^3}=D_x{}^3y, \qquad \text{etc.},$$

interpreting the left-hand sides of these equations, however, *not as ratios,* but as a *single, homogeneous* (and altogether clumsy!) *notation* for that which is expressed more simply by Cauchy's D.

Remark. The operator D_x shall be written when desired as $\dfrac{d}{dx}$. Thus

$$D_x\frac{x}{a-x} \qquad \text{appears as} \qquad \frac{d}{dx}\frac{x}{a-x}.$$

Again, the equation

$$D_x{}^2y=D_x(D_xy)$$

appears as

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{dy}{dx}.$$

Finally, the following notation is sometimes used:

$$\frac{d^2y}{dx}=D_x{}^2y\,dx, \qquad \frac{d^3y}{dx^2}=D_x{}^3y\,dx, \qquad \text{etc.}$$

4. Technique of Differentiation. Consider, for example, Formula II, Chapter II, § 6:

$$D_x(u+v)=D_xu+D_xv.$$

On writing this formula in terms of differentials, we have

$$\frac{d(u+v)}{dx}=\frac{du}{dx}+\frac{dv}{dx}.$$

Now multiply this equation through by dx:

$$d(u+v)=du+dv.$$

Hence the theorem: *The differential of the sum of two functions is equal to the sum of the differentials of these functions.*

The others of the General Formulas, Chapter II, §§ 6, 7, can be treated in a similar way and lead to corresponding theorems in differentials, embodied in the following important group of formulas.

General Formulas of Differentiation.

I. $$d(cu)=c\,du.$$

II. $$d(u+v)=du+dv.$$

III. $$d(uv)=u\,dv+v\,du.$$

IV. $$d\frac{u}{v}=\frac{v\,du-u\,dv}{v^2}.$$

As already explained, Theorem V reduces to an obvious algebraic identity:

$$\frac{du}{dx}=\frac{du}{dy}\frac{dy}{dx},$$

and so does not need to be tabulated.

Of the special formulas hitherto considered, only two need be tabulated, namely:

Special Formulas of Differentiation.

1. $$dc=0.$$

2. $$dx^n=nx^{n-1}dx.$$

The first of these formulas says that the differential of a constant is zero. The second is valid, not only when x is the independent variable, but when x is any function whatever of the independent variable, t. Thus if

(1) $$u=\sqrt{1-t}$$

and we set

(2) $$x = 1 - t,$$

equation (1) becomes

(3) $$u = x^{\frac{1}{2}}.$$

Hence $$du = \tfrac{1}{2} x^{-\frac{1}{2}} dx.$$

But $$dx = d1 + d(-t) = 0 - dt,$$

and thus $$du = \frac{-dt}{2\sqrt{1-t}} \quad \text{or} \quad \frac{du}{dt} = -\frac{1}{2\sqrt{1-t}}.$$

The student should copy off neatly on a card the size of a postal the General Formulas I–IV, the Special Formulas 1., 2., leaving room for a few further special formulas. All the differentiations of the elementary function of the Calculus are based on these two groups of formulas.

To *differentiate* a function means henceforth to find either its derivative or its differential. Of course, when one of these is known, the other can be found by merely multiplying or dividing by the differential of the independent variable.

We proceed to show by a few typical examples how differentials are used in differentiation.

Example 1. Let $$u = 12 - 5x + 7x^3.$$
To find du.

Take the differential of each side of this equation, and apply at the same time Formula II:

$$du = d(12) + d(-5x) + d(7x^3).$$

By Formula 1, $$d(12) = 0.$$

By Formula I,

$$d(-5x) = -5\,dx \quad \text{and} \quad d(7x^3) = 7\,dx^3.$$

Hence $$du = -5\,dx + 21x^2\,dx$$
$$= (-5 + 21x^2)\,dx$$

and $$\frac{du}{dx} = -5 + 21x^2.$$

These steps correspond precisely to the steps the student would take if he were using derivatives, only he would not have written them all out in detail. He would have written down at sight:

$$D_x u = -5 + 21x^2.$$

He can avail himself of the facility he has already acquired and shorten the work as follows. Since

$$du = D_x u\, dx,$$

he can begin by writing

$$du = (\qquad\qquad\qquad)dx,$$

and then fill in the parenthesis with the derivative.*

Example 2. Let $$u = \frac{a^2 - x^2}{a^2 + x^2}.$$

To find du.

By Formula IV we have:

$$du = \frac{(a^2 + x^2)d(a^2 - x^2) - (a^2 - x^2)d(a^2 + x^2)}{(a^2 + x^2)^2}$$

$$= \frac{(a^2 + x^2)(-2\,x\,dx) - (a^2 - x^2)(2\,x\,dx)}{(a^2 + x^2)^2}$$

$$= -\frac{4\,a^2x\,dx}{(a^2 + x^2)^2};$$

$$\frac{du}{dx} = -\frac{4\,a^2x}{(a^2 + x^2)^2}.$$

The student would probably prefer to work this example as follows. Remembering that

$$du = D_x u\, dx,$$

* The student must be careful not to omit any differentials. If one term of an equation has a differential as a factor, every term must have a differential as a factor. Such an equation as

$$du = -5 + 21\,x^2$$

is absurd, since the left-hand side is an infinitesimal and the right-hand not. Moreover, there is no such thing as $d_x u$.

begin by writing

$$du = \underline{\qquad\qquad\qquad\qquad} dx,$$

and then fill in the fraction by the old familiar methods of Chapter II.

In the two examples just considered, the processes with differentials correspond precisely to those with derivatives, with which the student is already familiar. This will always be true in any differentiation in which *composite functions* are not involved; *i.e.* whenever, according to our earlier methods, the vanished Theorem V of Chapter II, § 8 was not used. It is in the differentiation of composite functions that the method of differentials presents advantages over the earlier method. We turn in the next paragraphs to such examples.

EXERCISES

Differentiate each of the following functions by the method of differentials, and test the result by the methods of Chapter II.

1. $u = x^3 - 3x + 1.$ *Ans.* $du = 3x^2dx - 3dx.$

2. $y = a + bx + cx^2.$ *Ans.* $dy = b\,dx + 2\,cx\,dx.$

3. $w = a^3 - z^3.$ *Ans.* $dw = -3z^2dz.$

4. $s = 96t - 16t^2.$ *Ans.* $\dfrac{ds}{dt} = 96 - 32t.$

5. $s = v_0t + \frac{1}{2}gt^2.$ *Ans.* $\dfrac{ds}{dt} = v_0 + gt.$

6. $u = \dfrac{1-x}{1+x}.$ *Ans.* $du = \dfrac{-2\,dx}{(1+x)^2}.$

7. $y = \dfrac{x}{1+x^2}.$ *Ans.* $dy = \dfrac{dx - x^2dx}{(1+x^2)^2}.$

8. $z = \dfrac{1+x+x^2}{2x}.$ *Ans.* $dz = \dfrac{x^2-1}{2x^2}dx.$

9. $u = \dfrac{3-2x+x^3}{4+x^2-x^3}.$ **10.** $y = \dfrac{a^4-x^4}{a^4+a^2x^2+x^4}.$

5. Continuation. Differentiation of Composite Functions.

Example 3. Let $u = \sqrt{1 + x + x^2}$.

To find $\frac{du}{dx}$.

Here, we begin by computing du. To do this, introduce a new variable, y, setting

$$y = 1 + x + x^2.$$

Then
$$u = y^{\frac{1}{2}}.$$

Next, take the differential of each side of this equation. By Special Formula 2 above,

$$du = dy^{\frac{1}{2}} = \tfrac{1}{2} y^{-\frac{1}{2}}\, dy.$$

Moreover,
$$dy = (1 + 2\,x)dx.$$

Hence
$$du = \frac{(1 + 2\,x)dx}{2\sqrt{1 + x + x^2}}$$

and
$$\frac{du}{dx} = \frac{1 + 2\,x}{2\sqrt{1 + x + x^2}}.$$

Let the student carry through the above differentiation by the methods of Chapter II and compare his work step by step with the foregoing. He will find that, although the two methods are in substance the same, the method of differentials is simpler in form, *since no explicit use of Theorem* V *here is made.*

*Abbreviated Method.** The solution by differentials can be still further abbreviated by not introducing explicitly a new

* The student should not hasten to take this step himself. He will do well to omit the text that follows till he has worked a score or more of problems in differentiating composite functions as set forth under Example 3, introducing each time explicitly a new variable, as y, z, etc. Not until he comes himself to feel that the abbreviation is an aid, should he attempt to use it.

variable, y. The problem is to find du, when

$$u = (1 + x + x^2)^{\frac{1}{2}}.$$

Now, Special Formula 2, as has already been pointed out, holds, not merely when x is the independent variable, but for any *function* whatsoever. It might, for example, equally well be written in the form:

$$d[\phi(x)]^n = n[\phi(x)]^{n-1}\,d\phi(x).$$

In the present case, then, the *content* of that theorem, — the *essential and complete truth* it contains, — enables us to write down at once the equation:

$$d(1 + x + x^2)^{\frac{1}{2}} = \tfrac{1}{2}(1 + x + x^2)^{-\frac{1}{2}}\,d(1 + x + x^2).$$

This last differential is computed at sight, and thus the answer is obtained in two steps.

Even these two steps are carried out mentally as a single process, when the student has reached the highest point in the technique of differentiation. He then thinks of the formula:

$$d\sqrt{x} = \frac{dx}{2\sqrt{x}},$$

realizes that it holds, not merely when x is the independent variable, but for any function of x, and so writes down first the easy part of the right-hand side of the equation, thus:

$$d\sqrt{1 + x + x^2} = \frac{\qquad}{2\sqrt{1 + x + x^2}},$$

carrying in his head the fact that the numerator is the differential of the radicand, *i.e.* $d(1 + x + x^2)$. This differentiation he performs mentally, and thus has the final answer with no intermediate work on paper:

$$d\sqrt{1 + x + x^2} = \frac{(1 + 2x)dx}{2\sqrt{1 + x + x^2}}.$$

Example 4. The method of differentials is especially useful in the case of implicit functions. Thus, to find the derivative of y with respect to x when

$$x^3 - 3xy + 2y^4 = 1.$$

Take the differential of each side:

$$3x^2\,dx - 3x\,dy - 3y\,dx + 8y^3\,dy = 0.$$

Next, collect the terms in dx by themselves; the others will contain dy as a factor:

$$(3x^2 - 3y)dx + (8y^3 - 3x)dy = 0.$$

Hence $$\frac{dy}{dx} = \frac{3y - 3x^2}{8y^3 - 3x}.$$

EXERCISES

Differentiate the following twelve functions by the method of differentials and also by the methods of Chapter II (in either order), introducing each time *explicitly* the auxiliary variable, if one is used.

1. $u = \sqrt{a^4 + a^2x^2 + x^4}.$ *Ans.* $du = \dfrac{(a^2x + 2x^3)dx}{\sqrt{a^4 + a^2x^2 + x^4}}.$

2. $y = \dfrac{1}{\sqrt{1 - x^2}}.$ *Ans.* $dy = \dfrac{x\,dx}{(1 - x^2)^{\frac{3}{2}}}.$

3. $u = \dfrac{1}{1 - x}.$ *Ans.* $du = \dfrac{dx}{(1 - x)^2}.$

Suggestion. Introduce an auxiliary variable $y = 1 - x$. Then $u = y^{-1}$.

4. $u = \dfrac{1}{(1 - x)^2}.$ *Ans.* $\dfrac{du}{dx} = \dfrac{2}{(1 - x)^3}.$

5. $y = \dfrac{1}{1 + x^2}.$ *Ans.* $\dfrac{dy}{dx} = \dfrac{-2x}{(1 + x^2)^2}.$

6. $s = \dfrac{a^2}{(a+t)^2}$. *Ans.* $\dfrac{ds}{dt} = -\dfrac{2a^2}{(a+t)^3}$.

7. $2x^2 - xy + 4y^2 = 5$. *Ans.* $\dfrac{dy}{dx} = \dfrac{4x - y}{x - 8y}$.

8. $xy = a^2$. *Ans.* $\dfrac{dy}{dx} = -\dfrac{y}{x}$.

9. $y^2 = 2mx$. *Ans.* $\dfrac{dy}{dx} = \dfrac{m}{y}$.

10. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. *Ans.* $\dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y}$.

11. $2x^2 + 3y^2 = 10$. *Ans.* $\dfrac{dy}{dx} = -\dfrac{2x}{3y}$.

12. $2xy - x + y = 0$. *Ans.* $\dfrac{dy}{dx} = \dfrac{1 - 2y}{2x + 1}$.

The student can work the problems at the end of Chapter II by the method of differentials. For further practice, if desired, the following examples are appended.

13. $u = (x^2 + 1)\sqrt{x^3 - x}$. *Ans.* $\dfrac{du}{dx} = \dfrac{7x^4 - 2x^2 - 1}{2\sqrt{x^3 - x}}$.

14. $y = (x + 2b)(x - b)^2$. *Ans.* $\dfrac{dy}{dx} = 3(x^2 - b^2)$.

15. $u = \dfrac{x}{\sqrt{a^2 - x^2}}$. *Ans.* $\dfrac{du}{dx} = \dfrac{a^2}{\sqrt{(a^2 - x^2)^3}}$.

16. $u = \sqrt{\dfrac{a - x}{x}}$. *Ans.* $\dfrac{du}{dx} = -\dfrac{a}{2x\sqrt{ax - x^2}}$.

17. $u = \dfrac{x - a}{\sqrt{2ax - x^2}}$. *Ans.* $\dfrac{du}{dx} = \dfrac{a^2}{\sqrt{(2ax - x^2)^3}}$.

18. $u = \left(\dfrac{x^2 + a^2}{x}\right)^2$. *Ans.* $\dfrac{du}{dx} = 2\dfrac{x^3}{x^4 - a^4}$.

19. $z = \left(\dfrac{y^4 + b^4}{y^2}\right)^2$. *Ans.* $\dfrac{dz}{dy} = 4\dfrac{y^8 - b^8}{y^5}$.

20. $u = \dfrac{2x^2 + a^2}{x^3}\sqrt{a^2 - x^2}$. *Ans.* $\dfrac{du}{dx} = -\dfrac{3a^4}{x^4\sqrt{a^2 - x^2}}$.

21. $u = \sqrt{\dfrac{x^2 - x + 1}{x^2 + x + 1}}$. *Ans.* $\dfrac{du}{dx} = \dfrac{3a^4}{(x^2 + x + 1)\sqrt{x^4 + x^2 + 1}}$.

22. $u = \dfrac{(x - x^3)^{\frac{4}{3}}}{x^4}$. *Ans.* $\dfrac{du}{dx} = -\dfrac{8\sqrt[3]{x - x^3}}{3x^4}$.

23. $u = (x^{\frac{1}{3}} - a^{\frac{1}{3}})^4$. *Ans.* $\dfrac{du}{dx} = \dfrac{4(x^{\frac{1}{3}} - a^{\frac{1}{3}})^3}{3x^{\frac{2}{3}}}$.

24. $u = x(x^3 + 5)^{\frac{4}{3}}$. *Ans.* $\dfrac{du}{dx} = 5(x^3 + 1)(x^3 + 5)^{\frac{1}{3}}$.

25. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $\dfrac{dy}{dx} = -\sqrt[3]{\dfrac{y}{x}}$.

CHAPTER V

TRIGONOMETRIC FUNCTIONS

1. Radian Measure. In Trigonometry, the *radian measure* of an angle was introduced, apparently for no good purpose. The reason lies in the *importance for the Calculus* of this new system of measurement, and will become clear in the next paragraph, when we come to differentiate the sine. We will first recall the definition.

Let a circle be described with its centre at the vertex O of the angle; let r denote the length of the radius of the circle and s, that of the intercepted arc. Then the radian measure, θ, of the angle is defined as the ratio s/r:

(1) $$\theta = \frac{s}{r}.$$

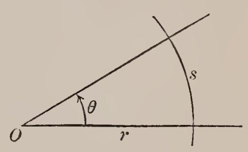

FIG. 34

For a right angle, $s = \frac{\pi r}{2}$, and hence $\theta = \frac{\pi}{2}$. A straight angle has the measure

$$\theta = \pi = 3.14159\ 26535\ 89793 \cdots.$$

Let ϕ be the measure of the given angle in degrees. Then θ and ϕ are proportional,

$$\theta = c\phi,$$

where c is a constant. To determine c, use a convenient angle whose measure is known in both systems; for example, a straight angle. For the latter,

$$\theta = \pi \qquad \text{and} \qquad \phi = 180.$$

Substituting these values in the above equation we find:

$$\pi = c\,180, \qquad c = \frac{\pi}{180},$$

and hence

(2) $$\theta = \frac{\pi}{180}\phi, \qquad \phi = \frac{180}{\pi}\theta.$$

This equation can also be written in the form

(3) $$\frac{\theta}{\pi} = \frac{\phi}{180}$$

and thus an easily remembered rule of conversion from radian measure to degree measure, or the opposite, obtained: *The radian measure of an angle is to* π *as its degree measure is to* 180.

The unit of angle in radian measure, *i.e.* the angle for which

$$\theta = 1 \qquad \text{and hence} \qquad s = r,$$

is called the *radian*. It is obvious geometrically that it is a little less than 60°. Its precise value (to hundredths of a second) is given by (2):

$$\phi\,|_{\theta=1} = \frac{180}{\pi} = 57°\,17'\,44.81'' \;(= 57.29578°).$$

On the other hand, the radian measure of an angle of 1° is

$$\theta\,|_{\phi=1} = \frac{\pi}{180} = .01745 \quad 32925 \quad 19943 \cdots.$$

The student should practice expressing the more important angles, as 30°, 45°, 60°, 90°, 120°, etc., in radian measure until he is thoroughly familiar with the new representation for them.

If, in particular, the radius of the circle is taken as unity, then θ and s are the same number:

(4) $$\theta = s, \qquad \text{when} \qquad r = 1;$$

or *the arc is equal to the angle.* Thus the radian measure of an angle might have been defined as the length of the intercepted

arc in the unit circle (*i.e.* the circle of unit radius with its centre at O).

Graph of $\sin x$. It is important for the student to make an accurately drawn graph of the function

$$y = \sin x,$$

x being taken in radian measure. Let the unit of length, as usual, be the same on both axes, and let it be chosen as 1 cm. For this purpose Peirce's Table of Integrals (the table of Trigonometric Functions near the end) is especially convenient, since the outside column gives the angles in radian measure, and thus as many points of the graph as are desired can be plotted directly from the tables.

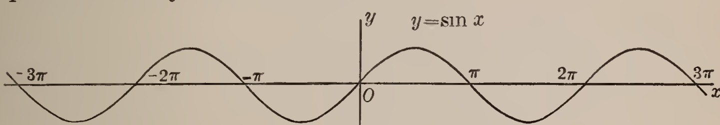

FIG. 35

Since $$\sin(\pi - x) = \sin x$$

each determination of the coordinates (x, y) of a point on the graph, for which $0 < x < \frac{\pi}{2}$ yields at once a second point, namely $(\pi - x, y)$. Thus one arch of the curve is readily constructed from the Tables.*

From this arch a templet, or curved ruler, is made as follows. Lay a card under the arch and with a needle prick through enough points so that the templet can be cut accurately with the scissors.

By means of the templet further arches can be drawn mechanically, and thus the curve is readily continued in both

* The graph could be made directly without tables from purely geometrical considerations. Draw a circle of unit radius. Construct geometrically convenient angles, as those obtained from a right angle by successive bisectors. Measure any one of these angles, $\measuredangle ABP_n$, in radians and this number will be the abscissa of the point on the graph, the

directions to the edges of the paper.* Put this curve in the upper quarter of a sheet of centimetre paper.

The graph brings out clearly the property of the function expressed by the word *periodic.* The function admits the *period* 2π, since

$$\sin(x + 2\pi) = \sin x$$

Graph of $\cos x$. By means of the templet the graph of the function

$$y = \cos x$$

can now be drawn mechanically. This function also admits the period 2π:

$$\cos(x + 2\pi) = \cos x.$$

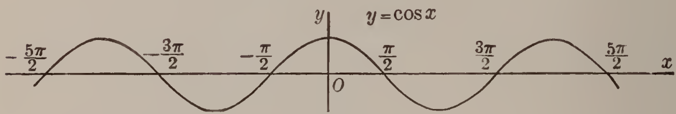

FIG. 37

ordinate being the perpendicular dropped from P_n on the line BA. Thus, if $n = 3$, the coordinates of the point on the graph are :

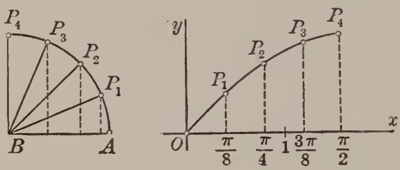

FIG. 36

$$x = \frac{3\pi}{8} = 1.18, \qquad y = .92.$$

A second point of the arch, that corresponding to P_5, has the same y, its coordinate being

$$x = \pi - \frac{3\pi}{8} = 1.96, \qquad y = .92.$$

Of course, the distance π must be laid off on the axis of x by measurement; it cannot be constructed geometrically from the unit length. This done, the further abscissae are found by successive bisectors.

* In order to obtain the most satisfactory figure, observe that the curve has a point of inflection at each of its intersections with the axis of x, the tangent there making an angle of $\pm 45°$ with that axis. Since a curve separates very slowly from an inflectional tangent, it will be well to draw these tangents with a ruler. On laying down the templet, the curve can then be ruled in from the latter with great accuracy. It will not separate sensibly from its tangent for a considerable distance from a point of inflection.

Put the graph in the second quarter of the sheet, choosing the axis of y for this curve in the same vertical line as the axis of y for the sine curve above. There remains the lower half of the sheet for the next graph.

Graph of $\tan x$. The same tables make it easy to plot points profusely on the graph of the function

$$y = \tan x$$

in the interval $0 \leqq x < \frac{\pi}{2}$. Take the axis of y in the same ver-

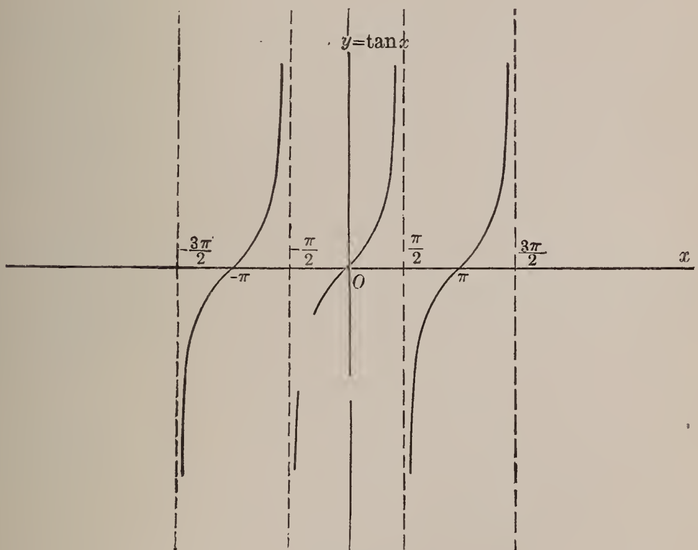

FIG. 38

tical line as in the case of the preceding graphs. This done, a second templet is made and by means of it the graph is drawn mechanically for values of x such that $-\frac{\pi}{2} < x < 0$.

It is desirable furthermore to plot the function in the two adjacent intervals

$$\frac{\pi}{2} < x < \frac{3\pi}{2}, \qquad -\frac{3\pi}{2} < x < -\frac{\pi}{2},$$

in order to suggest the fact that this function admits the period π:

$$\tan(x+\pi) = \tan x.$$

2. Differentiation of $\sin x$. To differentiate the function

(1) $$y = \sin x,$$

apply the definition of a derivative given in Chap. II, § 1. Give to x an arbitrary value x_0 and compute the corresponding value y_0 of y;

$$y_0 = \sin x_0.$$

Then give x an increment Δx, and compute again the corresponding value of y:

$$y_0 + \Delta y = \sin(x_0 + \Delta x).$$

Hence

$$\Delta y = \sin(x_0 + \Delta x) - \sin x_0,$$

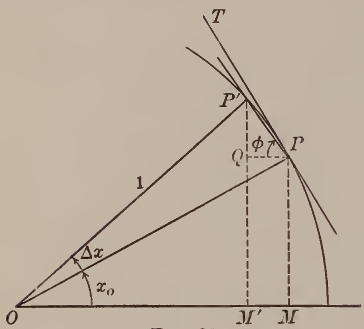

FIG. 39

(2) $$\frac{\Delta y}{\Delta x} = \frac{\sin(x_0 + \Delta x) - \sin x_0}{\Delta x}.$$

It is at this point in the process that the specific properties of the function $\sin x$ come into play. Here, the representation of $\sin x$ by means of the unit circle, familiar from the beginning of Trigonometry, is the key to the solution. From the figure it is clear that

$$\sin x_0 = MP, \qquad \sin(x_0 + \Delta x) = M'P',$$

$$\Delta y = \sin(x_0 + \Delta x) - \sin x_0 = QP', \qquad \Delta x = \overset{\frown}{PP'}.$$

Hence

(3) $$\frac{\Delta y}{\Delta x} = \frac{QP'}{\overset{\frown}{PP'}},$$

and so we want to know the limit approached by the latter ratio:

$$\lim_{P' \doteq P} \frac{QP'}{\stackrel{\frown}{PP'}}.$$

By virtue of the Fundamental Theorem of Chap. IV, § 2, we can replace this ratio by a simpler one, since the arc $\stackrel{\frown}{PP'}$ and the chord $\overline{PP'}$ are equivalent infinitesimals: *

$$\lim \frac{\overline{PP'}}{\stackrel{\frown}{PP'}} = 1.$$

Hence
$$\lim_{P' \doteq P} \frac{QP'}{\stackrel{\frown}{PP'}} = \lim_{P' \doteq P} \frac{QP'}{\overline{PP'}}.$$

On the other hand, the triangle QPP' is a triangle of reference for the $\measuredangle\, QPP' = \phi$, and so

$$\frac{QP'}{\overline{PP'}} = \sin \phi.$$

When P' approaches P, the secant PP' (*i.e.* the indefinite line determined by the two points P and P') approaches the tangent PT at P, and thus

$$\lim_{P' \doteq P} \phi = \measuredangle\, QPT = \frac{\pi}{2} - x_0.$$

Finally, then,

$$\lim_{P' \doteq P} \frac{QP'}{\overline{PP'}} = \lim_{P' \doteq P} \sin \phi = \sin\left(\frac{\pi}{2} - x_0\right) = \cos x_0,$$

and consequently
$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \cos x_0,$$

* The student should assure himself of the truth of this statement by visualizing the figure (making an accurate drawing with ruler and compass for angles of 30°, 15°, and $7\frac{1}{2}$°, the circle used being 10 in. in diameter) and realizing that, when P' is near P, the difference in length between the arc and the chord is but a minute per cent of the length of either one. A formal proof will be found below.

or, on dropping the subscript,

$$D_x \sin x = \cos x. \tag{4}$$

This theorem gives rise to the following theorem in differentials:

$$d \sin x = \cos x\, dx. \tag{5}$$

Reason for the Radian. The reason for measuring angles in terms of the radian as the unit now becomes clear. Had we used the degree, the increment Δx would not have been equal to $\overset{\frown}{PP'}$; we should have had:

$$\frac{\Delta x}{360} = \frac{\overset{\frown}{PP'}}{2\pi}, \qquad \text{or} \qquad \Delta x = \frac{180}{\pi} \overset{\frown}{PP'}.$$

Hence (3) would have read:

$$\frac{\Delta y}{\Delta x} = \frac{\pi}{180} \cdot \frac{QP'}{\overset{\frown}{PP'}},$$

and thus the formula of differentiation would have become:

$$D_x \sin x = \frac{\pi}{180} \cos x.$$

The saving of labor in not being obliged to multiply by this constant each time we differentiate is great. Still more important, however, is the elimination of a multiplier which is of the nature of an extraneous constant, whose presence would have obscured the essential simplicity of the formulas of the Calculus.

EXERCISE

Prove in a similar manner that

$$D_x \cos x = -\sin x.$$

3. Certain Limits. In the foregoing paragraph we have made use of the fact that *the ratio of the arc to the chord approaches* 1 *as its limit.* A formal proof of this theorem, based on the

axioms of geometry, can be given as follows. Draw the tangent at P and erect a perpendicular at P' cutting the tangent in Q. Denote the angle $\measuredangle\, P'PQ$ by α.

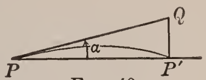

FIG. 40

Then

$$\overline{PP'} < \overset{\smile}{PP'} < PQ + P'Q;$$

for i) a straight line is the shortest distance between two points; and ii) a convex curved line is less than a convex broken line which envelops it and has the same extremities. But

$$PQ = \frac{\overline{PP'}}{\cos\alpha}, \qquad P'Q = \overline{PP'}\tan\alpha.$$

Hence

$$1 < \frac{\overset{\smile}{PP'}}{\overline{PP'}} < \frac{1}{\cos\alpha} + \tan\alpha.$$

When α approaches 0, the right-hand member of the double inequality approaches 1; hence the middle member must also approach 1, or

$$\lim \frac{\overset{\smile}{PP'}}{\overline{PP'}} = 1, \qquad \text{q.e.d.}$$

The foregoing proof holds, not merely for a circle, but for any curve with a convex arc $\overset{\smile}{PP'}$. Consequently the theorem is established generally.

The Limit $\lim\limits_{\alpha \doteq 0} \dfrac{\sin\alpha}{\alpha}$. From Fig. 41 it is clear that

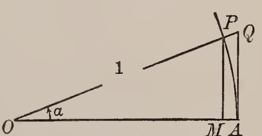

FIG. 41

$$MP = \sin\alpha, \qquad \overset{\smile}{AP} = \alpha,$$

and hence

$$\frac{\sin\alpha}{\alpha} = \frac{MP}{\overset{\smile}{AP}}.$$

By direct inspection of the figure it is seen, then, that

$$(1) \qquad \lim_{\alpha \doteq 0} \frac{\sin\alpha}{\alpha} = 1.$$

A formal proof of this equation can be given as follows. From Fig. 42

$$\overline{PP'} = 2\sin\alpha, \qquad \overset{\smile}{PP'} = 2\alpha.$$

Hence
$$\frac{\sin\alpha}{\alpha} = \frac{\overline{PP'}}{\overset{\smile}{PP'}},$$

and therefore, by the proposition just established,

$$\lim_{\alpha \doteq 0} \frac{\sin\alpha}{\alpha} = \lim_{P' \doteq P} \frac{\overline{PP'}}{\overset{\smile}{PP'}} = 1.$$

Another Proof of (1). The area of the sector OAP, Fig. 42, is $\frac{1}{2}\alpha$, and it obviously lies between the areas of the triangles OMP and OPN. Hence

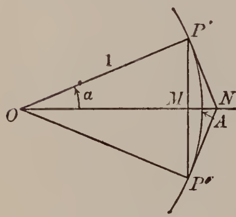

FIG. 42

$$\tfrac{1}{2}\sin\alpha\cos\alpha < \tfrac{1}{2}\alpha < \tfrac{1}{2}\tan\alpha$$

or
$$\cos\alpha < \frac{\alpha}{\sin\alpha} < \frac{1}{\cos\alpha}.$$

When α approaches 0, each of the extreme terms approaches 1, and so the middle term must also do so, q. e. d.

From Peirce's *Tables*, p. 130, we see that

$$\sin 4° \, 40' = .0814,$$

and the same angle, measured in radians, also has the value .0814, to three significant figures. Thus for values of α not exceeding .0814, $\sin\alpha$ differs from α by less than one part in 800, or one-eighth of one per cent.

The Limits $\lim\limits_{\alpha \doteq 0} \dfrac{1-\cos\alpha}{\alpha}$ *and* $\lim\limits_{\alpha \doteq 0} \dfrac{1-\cos\alpha}{\alpha^2}$. From Fig. 42, the first of these limits is seen to have the value 0:

$$\lim_{\alpha \doteq 0} \frac{1-\cos\alpha}{\alpha} = 0.$$

A formal proof can be derived at once by the method employed in the evaluation of the next limit,

$$\lim_{\alpha \doteq 0} \frac{1 - \cos \alpha}{\alpha^2}.$$

Expressing $1 - \cos \alpha$ in terms of the half angle, we have

$$1 - \cos \alpha = 2 \sin^2 \frac{\alpha}{2}.$$

Hence
$$\frac{1 - \cos \alpha}{\alpha^2} = \frac{2 \sin^2 \frac{\alpha}{2}}{\alpha^2} = \frac{1}{2} \left[\frac{\sin \frac{\alpha}{2}}{\frac{\alpha}{2}} \right]^2$$

and
$$\lim_{\alpha \doteq 0} \frac{1 - \cos \alpha}{\alpha^2} = \frac{1}{2} \lim_{\alpha \doteq 0} \left[\frac{\sin \frac{\alpha}{2}}{\frac{\alpha}{2}} \right]^2 = \frac{1}{2}.$$

EXERCISES

In the accompanying figure determine the following limits when α approaches 0:

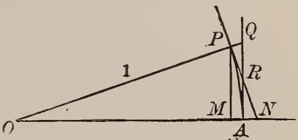

FIG. 43

1. $\lim \frac{AR}{MP}$. *Ans.* $\frac{1}{2}$.

2. $\lim \frac{AQ}{\overset{\frown}{AP}}$. *Ans.* 1.

3. $\lim \frac{RQ}{MP}$.

4. $\lim \frac{RP}{\overset{\frown}{AP}}$.

5. $\lim \frac{PN}{\overline{AP}}$.

6. $\lim \frac{MA}{PQ}$.

7. $\lim \frac{PQ}{AN}$.

8. $\lim \frac{RQ}{PN}$.

Determine the principal part of each of the following infinitesimals, referred to α as principal infinitesimal:

9. MP. *Ans.* α.

10. PR. *Ans.* $\frac{1}{2}\alpha$.

11. RQ.

12. PN.

13. AQ.

14. MA. *Ans.* $\frac{1}{2}\alpha^2$.

15. PQ.

16. MN.

17. $AQ - MP$.

4. Critique of the Foregoing Differentiation. The differentiation of $\sin x$ as given in § 1 has the advantage of being direct and lucid, and thus easily remembered. Each analytic step is mirrored in a simple geometric construction. It has the disadvantage, however, of incompleteness. For, first, we have allowed Δx, in approaching 0, to pass only through positive values; and secondly we have assumed x_0 to lie between 0 and $\frac{1}{2}\pi$. Hence there are in all seven more cases to consider.

An analytic method that is simple and at the same time general is the following. Recall the Addition Theorem for the sine:

$$\sin(a+b) = \sin a \cos b + \cos a \sin b,$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b,$$

whence $$\sin(a+b) - \sin(a-b) = 2\cos a \sin b.$$

Let $$a+b = x_0 + \Delta x, \qquad a-b = x_0.$$

Solving these last equations for a and b, we get:

$$a = x_0 + \frac{\Delta x}{2}, \qquad b = \frac{\Delta x}{2}.$$

Thus $$\sin(x_0+\Delta x) - \sin x_0 = 2\cos\left(x_0 + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2},$$

and the difference-quotient becomes

$$\frac{\Delta y}{\Delta x} = \cos\left(x_0 + \frac{\Delta x}{2}\right)\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}.$$

The first factor on the right approaches the limit $\cos x_0$ when Δx approaches 0. On setting $\frac{1}{2}\Delta x = \alpha$, the second factor becomes

$$\frac{\sin\alpha}{\alpha}.$$

Hence the factor approaches 1. Thus

$$\lim_{\Delta x \doteq 0}\frac{\Delta y}{\Delta x} = \cos x_0,$$

or, on dropping the subscript,

$$D_x \sin x = \cos x.$$

5. Differentiation of $\cos x$, $\tan x$, **etc.** To differentiate the function $\cos x$, introduce a new variable, y, by the equation

$$y = \frac{\pi}{2} - x. \qquad \text{Hence} \qquad x = \frac{\pi}{2} - y,$$

and
$$\cos x = \cos\left(\frac{\pi}{2} - y\right) = \sin y.$$

Taking the differential of each side of the equation thus obtained, we have:

$$d \cos x = d \sin y = \cos y\, dy.$$

But $\qquad \cos y = \sin x \qquad$ and $\qquad dy = -dx.$

Hence

(1) $$d \cos x = -\sin x\, dx.$$

To differentiate the function $\tan x$, set

$$\tan x = \frac{\sin x}{\cos x}.$$

Hence
$$d \tan x = \frac{\cos x\, d \sin x - \sin x\, d \cos x}{\cos^2 x}$$

$$= \frac{\cos^2 x\, dx + \sin^2 x\, dx}{\cos^2 x} = \frac{dx}{\cos^2 x},$$

and thus

(2) $$d \tan x = \sec^2 x\, dx.$$

It is shown in a similar manner (or by setting $x = \frac{\pi}{2} - y$ in the equation just deduced) that

(3) $$d \cot x = -\csc^2 x\, dx.$$

These are the important formulas of differentiation for the trigonometric functions. By means of them all other differentiations of these functions can be readily performed. Thus,

to differentiate the function $\sec x$, set

$$\sec x = (\cos x)^{-1}.$$

Then
$$d \sec x = -\frac{d \cos x}{\cos^2 x} = \frac{\sin x \, dx}{\cos^2 x}.$$

It is not desirable to tabulate the result, since one rarely has occasion to differentiate either $\sec x$ or $\csc x$, and when the occasion does arise, the differentiation can be worked out directly, as above.

The student should now add to his card of Special Formulas the four main formulas just obtained. This card will now read as follows:

1. $$dc = 0.$$
2. $$d\,x^n = nx^{n-1}\,dx.$$
3. $$d \sin x = \cos x \, dx.$$
4. $$d \cos x = -\sin x \, dx.$$
5. $$d \tan x = \sec^2 x \, dx.$$
6. $$d \cot x = -\csc^2 x \, dx.$$

6. Shop Work. To acquire facility in the use of the new results, the student should work a generous number of simple examples, for which the following are typical.

Example 1. To differentiate the function

$$u = \sin ax.$$

Let
$$y = ax.$$

Then
$$u = \sin y,$$

and
$$du = d \sin y = \cos y \, dy.$$

But
$$dy = a\,dx.$$

Hence, substituting, we have

$$du = a \cos ax \, dx \qquad \text{or} \qquad \frac{d}{dx} \sin ax = a \cos ax.$$

The solution can be abbreviated as follows. The equation

$$d \sin x = \cos x \, dx$$

is true, not merely when x is the independent variable. It holds, for example, in the form

$$d \sin y = \cos y \, dy,$$

where y is any function of x. Hence we can write immediately

$$d \sin ax = \cos ax \; d(ax),$$

and thus obtain the result

$$d \sin ax = a \cos ax \, dx.$$

Example 2. To differentiate the function

$$u = \sqrt{1 - k^2 \sin^2 \phi}.$$

Let $$z = 1 - k^2 \sin^2 \phi.$$

Then $$u = z^{\frac{1}{2}}$$

$$du = d z^{\frac{1}{2}} = \tfrac{1}{2} z^{-\frac{1}{2}} dz;$$

$$dz = - k^2 d \sin^2 \phi.$$

Let $$y = \sin \phi.$$

Then $$dy = \cos \phi \, d\phi$$

and $$d \sin^2 \phi = d y^2 = 2 y \, dy = 2 \sin \phi \cos \phi \, d\phi.$$

Hence $$du = \tfrac{1}{2} z^{-\frac{1}{2}}(- 2 k^2 \sin \phi \cos \phi \, d\phi)$$

or $$\frac{du}{d\phi} = - \frac{k^2 \sin \phi \cos \phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

Example 3. If $\sin x + \sin y = x - y,$

to find $\frac{dy}{dx}$. Take the differential of each side of the equation:

$$\cos x \, dx + \cos y \, dy = dx - dy.$$

Hence $$(\cos x - 1)dx + (\cos y + 1)dy = 0$$

and $$\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos y}.$$

EXERCISES

Differentiate the following functions.

1. $u = \cos ax.$ $\qquad \dfrac{du}{dx} = -a \sin ax.$

2. $y = \cos^2 x.$ $\qquad \dfrac{dy}{dx} = -2 \sin x \cos x.$

3. $y = \csc x.$ $\qquad \dfrac{dy}{dx} = \csc^2 x \cos x.$

4. $u = \tan \dfrac{x}{2}.$ $\qquad \dfrac{du}{dx} = \dfrac{1}{2} \sec^2 \dfrac{x}{2}.$

5. $u = \cot 2x.$ $\qquad \dfrac{du}{dx} = -2 \csc^2 2x.$

6. $u = \sec 3x.$

7. $u = \tan^2 ax.$

8. $u = \sin^3 x.$

9. $u = 1 - \sin x.$

10. $u = x + \tan x.$

11. $u = \cos^3 x.$

12. $u = \sec^2 x.$

13. $u = \sin x \cos x.$

14. $u = \dfrac{\sin x}{1 - \cos x}.$ $\qquad \dfrac{du}{dx} = -\dfrac{1}{2} \csc^2 \dfrac{x}{2}.$

15. $u = \sqrt{1 + \cos x}.$ $\qquad \dfrac{du}{dx} = -\dfrac{1}{\sqrt{2}} \sin \dfrac{x}{2}.$

16. $u = \dfrac{1 - \cos x}{1 + \cos x}.$

17. $u = \dfrac{1 + \sin x}{1 - \sin x}.$

18. $u = \dfrac{\sin x}{a + b \cos x}.$

19. $u = \dfrac{1}{a \cos x + b \sin x}.$

20. $u = \dfrac{1}{\sin x + \cos x}.$

21. $u = \dfrac{1}{(a + b \cos x)^2}.$

22.* $u = \text{vers}\, x.$ $\qquad \dfrac{du}{dx} = \sin x.$

23.* $u = \text{covers}\, x.$ $\qquad \dfrac{du}{dx} = -\cos x.$

* The *versed sine* and the *coversed sine* are defined as follows:

$$\text{vers}\, x = 1 - \cos x; \qquad \text{covers}\, x = 1 - \sin x.$$

24. $u = x \sin 2x.$

25. $u = \dfrac{\cos \frac{x}{2}}{x}.$

26. $u = \tan\left(\frac{\pi}{4} - \frac{x}{2}\right).$

27. $u = \cot\left(\frac{x}{2} - \frac{\pi}{4}\right).$

28. $u = \tan \dfrac{x}{1-x}.$

29. $u = \dfrac{\sin \pi x}{x}.$

30. $u = \sin x + \cos 2x.$

31. $u = x^2 \cos \pi x.$

32. $u = \dfrac{1}{\sqrt{1-k^2 \sin^2 \phi}}.$

33. $u = \dfrac{\cos \phi}{\sqrt{1-k^2 \sin^2 \phi}}.$

34. $x \cos y = \sin(x+y).$ $\quad \dfrac{dy}{dx} = -\dfrac{\cos(x+y) - \cos y}{\sin(x+y) + x \sin y}.$

35. $\tan x - \cot y = \sin x \sin y.$

36. $\sin x + \sin y = 1.$

37. $\tan \theta + \tan \phi = 2 \tan \phi \tan \theta.$

38. $x = y \sin y.$

7. Maxima and Minima. By means of the new functions studied in this chapter the range of problems in maxima and minima which can be treated by the Calculus has been materially enlarged. No new principles are involved; the student should go over carefully the paragraphs of Chap. III relating to this subject, before he proceeds farther with the present paragraph.

Example 1. A man in a rowboat 1 mile off shore wishes to go to a point which is 2 miles inland and 4 miles up the beach. If he can row at the rate of 5 miles an hour, but can walk only 3 miles an hour after he lands, in what direction should he row in order to get to his destination in the shortest possible time?

In the first place, it is clear that the straight line AEB is *not* the best path. For, if he rows toward a point P slightly farther up the beach, the amount by which he lengthens the leg AP of his path is very nearly equal to the amount by which

he shortens the leg PB.* Consequently the time is shortened.

On the other hand, P obviously ought not to be taken so far up the beach as D. The minimum occurs, therefore, for some intermediate point.

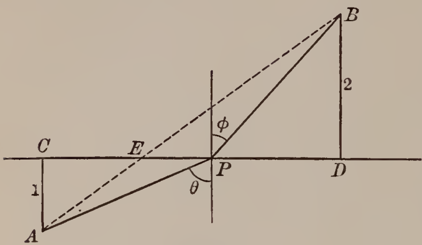

FIG. 44

Let the angles θ, ϕ be taken as indicated in the figure. Then, since $t = \dfrac{s}{v}$,

$$\text{time from } A \text{ to } P = \frac{AP}{5} = \frac{1}{5\cos\theta};$$

$$\text{time from } P \text{ to } B = \frac{PB}{3} = \frac{2}{3\cos\theta}.$$

Hence the total time, u, which is to be made a minimum is

$$(1) \qquad u = \frac{1}{5\cos\theta} + \frac{2}{3\cos\phi}.$$

Moreover, θ and ϕ are connected with each other by a relation which is readily obtained by expressing the distance CD in two ways:

$$(2) \qquad \tan\theta + 2\tan\phi = 4.$$

We are now ready to compute $du/d\theta$ and set it equal to 0:

$$du = -\frac{d\cos\theta}{5\cos^2\theta} - \frac{2\,d\cos\phi}{3\cos^2\phi}$$

$$= \frac{\sec^2\theta\sin\theta}{5}\,d\theta + \frac{2\sec^2\phi\sin\phi}{3}\,d\phi;$$

$$(3) \qquad \frac{du}{d\theta} = \frac{\sec^2\theta\sin\theta}{5} + \frac{2\sec^2\phi\sin\phi}{3}\,\frac{d\phi}{d\theta}.$$

* Let the student not leave this statement till he is absolutely convinced of its truth. An accurate figure on a large scale will bring the fact out clearly.

On setting $du/d\theta = 0$, we obtain the equation:

(4) $$\frac{\sec^2\theta\sin\theta}{5} = -\frac{2\sec^2\phi\sin\phi}{3}\frac{d\phi}{d\theta}.$$

Next, differentiate (2):

$$\sec^2\theta d\theta + 2\sec^2\phi d\phi = 0,$$

or

(5) $$\sec^2\theta = -2\sec^2\phi\frac{d\phi}{d\theta}.$$

Now, divide equation (4) by equation (5):*

(6) $$\frac{\sin\theta}{5} = \frac{\sin\phi}{3} \quad \text{or} \quad \frac{\sin\theta}{\sin\phi} = \frac{5}{3}.$$

The result, stated in words, is as follows: $\sin\theta$ *is to* $\sin\phi$ *as the velocity in water is to the velocity on land.*

Let the student work the general problem, in which all the data are taken in literal form, and verify the general result just stated.

In order actually to determine θ, equations (2) and (6) must be solved as simultaneous:

(7) $$\begin{cases} \tan\theta + 2\tan\phi = 4, \\ 3\sin\theta = 5\sin\phi. \end{cases}$$

This is done best by the method of Trial and Error, as it is called in Physics; Successive Approximations being the name usually given to it in Mathematics. It is a most important method in both sciences, and the student should let no opportunity go by to use the method whenever, as here, he meets a case which calls for it. Cf. Chap. VII, § 5.

The Corresponding Problem in Optics. We have stated and solved a problem which is not lacking in interest, but which appears to have no scientific importance. This very problem, however, occurs in Optics. The velocity of light is different in

* *i.e.* divide the left-hand side of (4) by the left-hand side of (5) for a new left-hand side; and do the same thing for the right-hand sides.

different media, such as air and water. Suppose two media to be in contact with each other, the common boundary being a plane. Let A be a luminous point, from which rays emanate in all directions. When the rays strike the bounding surface, they are all refracted and enter the second medium in case the velocity of light in that medium is less than in the first. One of the refracted rays will pass through a given point B. And now the law of light is that the time required for the light to pass from A to B is less for this path than for any other possible path.

If, then, the velocity of light in the first medium is u * and in the second medium, v, we have:

$$\frac{\sin \theta}{\sin \phi} = \frac{u}{v} = n,$$

where n is the *index of refraction* for the passage from the first medium into the second.

EXERCISES

1. A wall 27 ft. high is 64 ft. from a house. Find the length of the shortest ladder that will reach the house if one end rests on the ground outside the wall.

Take the angle which the ladder makes with the horizontal as the independent variable.

2. The equal sides of an isosceles triangles are each 8 in. long, the base being variable. Show that the triangle of maximum area is the one which has a right angle.

Take one of the base angles as the independent variable, ϕ.

3. A gutter is to be made out of a long strip of copper 9 in. wide by bending the strip along two lines parallel to the edges and distant respectively 3 in. from an edge. Thus the cross-section will be a broken line, made up of three straight lines, each 3 in. long. How wide should the gutter be at the

* The letter u used here has nothing to do with the u used above in solving the problem.

top, in order that its carrying capacity may be as great as possible? *Ans.* 6 in.

4. Johnny is to have a piece of pie, the perimeter of which is to be 12 in. If Johnny may choose the plate on which the pie is to be baked, what size plate would he naturally select?

5. A can-buoy in the form of a double cone is to be made from two equal circular iron plates by cutting out a sector from each plate and bending up the plate. If the radius of each plate is a, find the radius of the base of the cone when the buoy is as large as possible. *Ans.* $a\sqrt{\frac{2}{3}}$.

6. From a circular piece of filter paper a sector is to be cut and then bent into the form of a cone of revolution. Show that the largest cone will be obtained if the angle of the sector is .8165 of four right angles.

7. Two solid spheres, whose diameters are 8 in. and 18 in., have their centres 35 in. apart. At what point in their line of centres and between the spheres should a light be placed in order to illuminate the largest amount of spherical surface?

Ans. 8 in. from the centre of the smaller sphere.

8. Find the most economical proportions for a conical tent.

9. A block of stone is to be drawn along the floor by a rope. Find the angle which the rope should make with the horizontal in order that the tension may be as small as possible.

Ans. The angle of friction.

10. A block of stone is to be drawn up an inclined plane by a rope. Find the angle which the rope should make with the plane, in order that the tension in the rope be as small as possible.

11. A statue ten feet high stands on a pedestal that is 50 ft. high. How far ought a man whose eyes are 5 ft. above the ground to stand from the pedestal in order that the statue may subtend the greatest possible angle?

12. A steel girder 25 ft. long is moved on rollers along a passageway 12.8 ft. wide, and into a corridor at right angles

to the passageway. Neglecting the horizontal width of the girder, find how wide the corridor must be in order that the girder may go round the corner. *Ans.* 5.4 ft.

13. A gutter whose cross-section is an arc of a circle is to be made by bending into shape a strip of copper. If the width of the strip is a, find the radius of the cross-section when the carrying capacity of the gutter is a maximum. *Ans.* a/π.

14. A long strip of paper 8 in. wide is cut off square at one end. A corner of this end is folded over on to the opposite side, thus forming a triangle. Find the area of the smallest triangle that can thus be formed.

15. In the preceding question, when will the length of the crease be a minimum?

16. The captain of a man-of-war saw, one dark night, a privateersman crossing his path at right angles and at a distance ahead of c miles. The privateersman was making a miles an hour, while the man-of-war could make only b miles in the same time. The captain's only hope was to cross the track of the privateersman at as short a distance as possible under his stern, and to disable him by one or two well-directed shots; so the ship's lights were put out and her course altered in accordance with this plan. Show that the man-of-war crossed the privateersman's track $\frac{c}{b}\sqrt{a^2 - b^2}$ miles astern of the latter.

If $a = b$, this result is absurd. Explain.

17. The illumination of a small plane surface by a luminous point is proportional to the cosine of the angle between the rays of light and the normal to the surface, and inversely proportional to the square of the distance of the luminous point from the surface. At what height on the wall should an arc light be placed in order to light most brightly a portion of the floor a ft. distant from the wall?

Ans. About $\frac{7}{10}a$ ft. above the floor.

18. A town A situated on a straight river, and another town B, a miles farther down the river and b miles back from the river, are to be supplied with water from the river pumped by a single station. The main from the waterworks to A will cost \$ m per mile and the main to B will cost \$ n per mile. Where on the river-bank ought the pumps to be placed?

19. A telegraph pole 25 ft. high is to be braced by a stay 20 ft. long, one end of the stay being fastened to the pole and the other end to a short stake driven into the ground. How far from the pole should the stake be located, in order that the stay be most effective?

20. Into a full conical wine-glass whose depth is a and generating angle α there is carefully dropped a spherical ball of such a size as to cause the greatest overflow. Show that the radius of the ball is

$$\frac{a \sin \alpha}{\sin \alpha + \cos 2\alpha}.$$

21. A foot-ball field $2a$ ft. long and $2b$ ft. broad is to be surrounded by a running track consisting of two straight sides (parallel to the length of the field) joined by semicircular ends. The track is to be $4c$ ft. long. Show how it should be made in order that the shortest distance between the track and the foot-ball field may be as great as possible.

22.* The number of ems (or the number of sq. cms. of text) on this page and the breadths of the margins being given, what ought the length and breadth of the page to be that the amount of paper used may be as small as possible?

23. Assuming that the values of diamonds are proportional, other things being equal, to the squares of their weights, and that a certain diamond which weighs one carat is worth \$ m, show that it is safe to pay at least \$ $8m$ for two diamonds which together weigh 4 carats, if they are of the same quality as the one mentioned.

* Exs. 22–25 do not involve Trigonometry.

24. When a voltaic battery of given electromotive force (E volts) and given internal resistance (r ohms) is used to send a steady current through an external circuit of R ohms resistance, an amount of work, W, equivalent to

$$\frac{E^2R}{(r+R)^2} \times 10^7 \text{ ergs}$$

is done each second in the outside circuit. Show that, if different values be given to R, W will be a maximum when $R = r$.

25. An ice cream cone is to hold one-eighth of a pint. The slant height is l, and half the angle at the vertex is x. Find the value of x that will make the cost of manufacture of the cone a minimum. (*Ans.* $x = 35°.27.$)

8. Tangents in Polar Coordinates. Let

$$r = f(\theta)$$

be the equation of a curve in polar coordinates. We wish to find the direction of its tangent. The direction will be known if we can determine the angle ψ between the radius vector produced and the tangent. Let P, with the coordinates (r_0, θ_0), be an arbitrary point of the curve and $P' : (r_0 + \Delta r,\ \theta_0 + \Delta\theta)$ a neighboring point. Draw the chord PP' and denote the $\angle\ OP'P$ by ψ'. Then obviously

$$\lim_{P' \doteq P} \psi' = \psi_0.$$

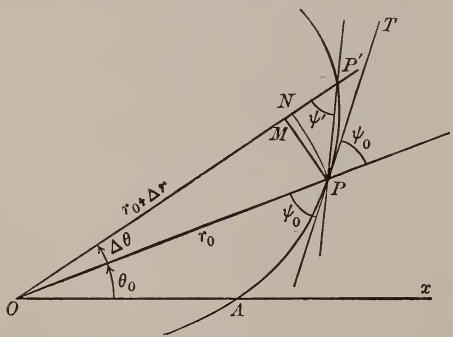

FIG. 45

To determine ψ_0, drop a perpendicular PM from P on the radius vector OP' and draw an arc PN of a circle with O as centre. The right triangle $MP'P$ is a triangle of refer-

ence for the angle ψ' and

$$\cot\psi' = \frac{P'M}{MP}.$$

Hence

(1) $$\cot\psi_0 = \lim_{P' \doteq P} \cot\psi' = \lim_{P' \doteq P} \frac{P'M}{MP}.$$

In the latter ratio we can replace $P'M$ and MP by more convenient infinitesimals; cf. Chap. IV, § 2. We observe that

$$MP = r_0 \sin\Delta\theta; \qquad \text{hence} \qquad \lim_{\Delta\theta \doteq 0} \frac{MP}{r_0\Delta\theta} = \lim_{\Delta\theta \doteq 0} \frac{\sin\Delta\theta}{\Delta\theta} = 1;$$

i.e. MP and $r_0\Delta\theta$ are equivalent infinitesimals.

Furthermore, $P'M$ and $P'N = \Delta r$ are also equivalent infinitesimals. For

$$P'M = P'N + NM$$

and

$$NM = r_0 - r_0 \cos\Delta\theta.$$

Hence

$$\frac{NM}{\Delta r} = \frac{r_0 \dfrac{1 - \cos\Delta\theta}{\Delta\theta}}{\dfrac{\Delta r}{\Delta\theta}}.$$

Now, by § 3,

$$\lim_{\Delta\theta \doteq 0} \frac{1 - \cos\Delta\theta}{\Delta\theta} = 0.$$

On the other hand,

$$\lim_{\Delta\theta \doteq 0} \frac{\Delta r}{\Delta\theta} = D_\theta r,$$

and this quantity is not, in general, 0. Hence

$$\lim_{\Delta\theta \doteq 0} \frac{NM}{\Delta r} = 0.$$

Returning to equation (1) we can now write the last limit in the form:

$$\lim_{P' \doteq P} \frac{P'M}{MP} = \lim_{\Delta\theta \doteq 0} \frac{\Delta r}{r_0\Delta\theta} = \frac{1}{r_0} D_\theta r;$$

or, dropping subscripts,

(2) $$\cot\psi = \frac{1}{r} D_\theta r.$$

In terms of differentials, this result can be written in either of the two forms:

$$(3) \qquad \cot\psi = \frac{dr}{r\,d\theta}, \qquad \tan\psi = \frac{r\,d\theta}{dr}.$$

Example. Consider the parabola in polar form:

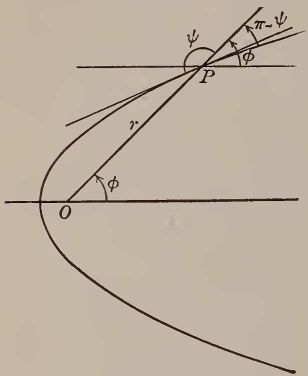

FIG. 46

$$r = \frac{m}{1-\cos\phi}$$

To determine ψ. Here,

$$dr = -\frac{m\sin\phi\,d\phi}{(1-\cos\phi)^2}.$$

Hence

$$\cot\psi = -\frac{m\sin\phi\,d\phi}{(1-\cos\phi)^2}\cdot\frac{1-\cos\phi}{m d\phi}$$

$$= -\frac{\sin\phi}{1-\cos\phi}.$$

In particular, at the extremity of the latus rectum, we have:

$$\cot\psi\,\Big|_{\phi=\frac{\pi}{2}} = -1, \qquad \psi = \frac{\pi}{4}+\pi,$$

and thus we obtain anew the result that the tangent there makes an angle of 45° with the axis of the parabola.

Again, at the vertex,

$$\cot\,\Big|_{\phi=\pi} = 0, \qquad \psi = \frac{\pi}{2},$$

and the tangent there is verified as perpendicular to the axis.

From the above equation,

$$\cot\psi = -\frac{\sin\phi}{1-\cos\phi},$$

a simple relation between ψ and ϕ can be deduced. Since

$$\frac{\sin\phi}{1-\cos\phi} = \frac{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{2\sin^2\frac{\phi}{2}} = \cot\frac{\phi}{2},$$

it follows that
$$\cot\psi = -\cot\frac{\phi}{2}.$$

But, for any angle, x,
$$\cot(\pi - x) = -\cot x.$$

Setting $x = \psi$ in the above equation, we have:
$$\cot(\pi - \psi) = \cot\frac{\phi}{2}.$$

Hence *
$$\pi - \psi = \frac{\phi}{2},$$

or, *the supplement of* ψ *is equal to* $\frac{\phi}{2}$. Thus we have a new proof of the familiar property of the parabola, that the tangent at any point P of the curve bisects the angle between the focal radius, OP, and a parallel to the axis, drawn through P.

EXERCISES

1. Plot the spiral,
$$r = \theta,$$
and show that the angle at which it crosses the prime direction when $\theta = 2\pi$ is $80° \, 57'$.

2. Plot the spiral,
$$r = \frac{1}{\theta}.$$
Show that it has an asymptote parallel to the prime vector.

Suggestion. Consider the distance of a point P of the curve from the prime direction, and find the limit of this distance when θ approaches 0.

Determine the angle at which the radius vector corresponding to $\theta = \pi/2$ meets this curve.

3. Plot the cardioid,
$$r = a(1 - \cos\phi),$$

* The trigonometric equation admits a second solution, namely, $(\pi - \psi) + \pi = \phi/2$. If, however, we agree to take ϕ and ψ so that $0 \leqq \phi < 2\pi$ and $0 \leqq \psi < \pi$, this second solution is ruled out.

and show that
$$\cot\psi = \frac{\sin\phi}{1-\cos\phi}.$$

At what angle is the curve cut by a line through the cusp perpendicular to the axis?

4. Prove that, for the cardioid,
$$\psi = \frac{\phi}{2}.$$

5. Show that the tangent to the cardioid is parallel to the axis of the curve when $\phi = \frac{2}{3}\pi$.

6. At what points of the cardioid is the tangent perpendicular to the axis of the curve?

7. Determine the rectangle which circumscribes the cardioid and has two of its sides parallel to the axis of the curve.

8. Show that, for the lemniscate,
$$r^2 = a^2 \cos 2\theta,$$
the angle ψ is given by the equation:
$$\cot\psi = -\tan 2\theta.$$
Hence, show that
$$\psi = \frac{\pi}{2} + 2\theta.$$

9. At what points of the lemniscate is the tangent parallel to the axis * of the curve?

Ans. At the point for which $\theta = \pi/6$, and the points which correspond to it by symmetry.

10. The points of the curve
$$r = f(\phi),$$
at which the tangent is parallel to the prime vector, are evidently those for which
$$y = r\sin\phi,$$

* The *axis* of any curve is a line of symmetry. The lemniscate has two such lines. The axis referred to in the text is that one of these lines which passes through the vertices of the curve.

considered as a function of ϕ through the mediation of the equation of the curve, has a maximum, a minimum, or a certain point of inflection. For these points, then,

$$\frac{dy}{d\phi} = r \cos \phi + \sin \phi \frac{dr}{d\phi} = 0.$$

Show that this condition is equivalent to the one used above in the special cases considered, namely:

$$\psi + \phi = \pi.$$

11. Plot the curve, $\quad r = a \cos 2\theta,$

taking $a = 5$ cm. Show that for this curve

$$\cot \psi = -2 \tan 2\theta.$$

12. At what points of the curve of question 11 is the tangent parallel to the axis?

Ans. For one of the points, $\tan \theta = \frac{1}{\sqrt{5}}$.

13. Plot the curve, $\quad r = a \cos 3\theta,$

taking $a = 5$ cm. Show that

$$\cot \psi = -3 \tan 3\theta.$$

14. At what points of the curve of question 13 is the tangent parallel to the axis of the lobe?

Ans. For one of these points, $\tan \theta = \sqrt{1 + \frac{2}{\sqrt{3}}}$.

15. The equation $\quad r = \frac{m}{1 + \sin \phi}$

represents a parabola referred to its focus as pole. Give a direct proof that the tangent to this curve at any point bisects the angle formed by the focal radius drawn to this point and a parallel to the axis through the point.

16. Show that the tangent to the hyperbola

$$r = \frac{m}{1 - \sqrt{3} \cos \phi}$$

at the extremity of the latus rectum makes an angle of 60° with the transverse axis.

17. Prove that the tangent to the ellipse

$$r = \frac{\mu}{\sqrt{3} - \cos\phi}$$

at the extremity of the latus rectum makes an angle of 30° with the major axis.

9. Differential of Arc. Let

$$y = f(x) \tag{1}$$

be the equation of a given curve. Let P, with the coordinates (x, y), be a variable point, and A a fixed point of the curve. Denote the length of the arc AP by s. Then s is a function of x; for, when x is given, we know P and thus s.

It is possible to determine the derivative of s, $D_x s$, as follows. By the Pythagorean Theorem we have (Chap. IV, Fig. 33),

$$\overline{PP'}^2 = \Delta x^2 + \Delta y^2.$$

Hence
$$\left(\frac{\overline{PP'}}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2.$$

Let Δx approach 0 as its limit. Then

$$\lim_{\Delta x \doteq 0}\left(\frac{\overline{PP'}}{\Delta x}\right)^2 = 1 + \lim_{\Delta x \doteq 0}\left(\frac{\Delta y}{\Delta x}\right)^2 = 1 + (D_x y)^2.$$

Since by § 3 the chord $\overline{PP'}$ and the arc $\overset{\frown}{PP'} = \Delta s$ are equivalent infinitesimals, it follows from the Fundamental Theorem of Chap. IV, § 2 that, in the above equation, $\overline{PP'}$ can be replaced by Δs. Hence

$$\lim_{\Delta x \doteq 0}\left(\frac{\overline{PP'}}{\Delta x}\right)^2 = \lim_{\Delta x \doteq 0}\left(\frac{\Delta s}{\Delta x}\right)^2 = (D_x s)^2,$$

and consequently

$$(D_x s)^2 = 1 + (D_x y)^2. \tag{2}$$

On replacing the derivatives in (2) by their values in terms of differentials, we have

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

or

$$ds^2 = dx^2 + dy^2. \tag{3}$$

This formula is easily interpreted geometrically by means of the triangle PMQ, Fig. 33. Since

$$PM = dx \qquad \text{and} \qquad MQ = dy,$$

it follows from the Pythagorean Theorem that

$$PQ = ds. \tag{4}$$

It is obvious geometrically that ds and Δs differ from each other by an infinitesimal of higher order; *i.e.* that they are equivalent infinitesimals.*

Formulas for $\sin\tau$, $\cos\tau$. From the triangle PMQ we can write down two further formulas:

$$\sin\tau = \frac{dy}{ds}, \qquad \cos\tau = \frac{dx}{ds}. \tag{5}$$

These formulas presuppose a suitable choice of τ. As s increases, the point P describes the curve in a definite sense. Let this be chosen as the positive sense of the tangent line at P. Then τ shall be the angle between the positive axis of x and this line. If τ were taken as the angle which the oppositely directed tangent makes with the positive axis of x, the $-$ sign must be written before each right-hand side in (5).

The formulas (5) suggest that x and y can be taken as functions of s:

$$x = \phi(s), \qquad y = \psi(s).$$

* In case the coordinates x and y are expressed as functions of a third variable t, dx will not in general be equal to Δx, but will differ from it by an infinitesimal of higher order. The triangle PMQ will then be replaced by a similar triangle PM_1Q_1, in which M_1 lies on the line PM, its distance from M being an infinitesimal of higher order.

This is, of course, always possible, since, when s is given, P, and hence also x and y, are determined.

Since

(6) $$ds = \pm\sqrt{dx^2 + dy^2},$$

we have from (5)

(7) $$\sin\tau = \pm\frac{dy}{\sqrt{dx^2+dy^2}}, \qquad \cos\tau = \pm\frac{dx}{\sqrt{dx^2+dy^2}},$$

no matter what choices of s and τ are made.* Furthermore,

(8) $$\sin\tau = \pm\frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}, \qquad \cos\tau = \pm\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}.$$

Which sign is to be used in (8) depends on which of the two possible determinations has been chosen for τ. Thus τ in a given case might be $30°$ or $30° + 180° = 210°$. If the first choice were made, $\tau = 30°$, then $\sin\tau$, $\cos\tau$, and $dy/dx = \tan\tau$ would all be positive quantities, and hence the upper signs must be taken. But if the other choice, $\tau = 210°$, is made, then $\sin\tau$ and $\cos\tau$ are negative, and the lower signs hold.

Example. Consider the parabola

$$y = x^2.$$

Let P be a point of the curve which lies in the first quadrant. Since

$$\tan\tau = \frac{dy}{dx} = 2x$$

is here positive, τ may be taken as an angle of the first quadrant. In that case, formulas (8) give

$$\sin\tau = \frac{2x}{\sqrt{1+4x^2}}, \qquad \cos\tau = \frac{1}{\sqrt{1+4x^2}}.$$

* The signs in (6) and (7) are not necessarily the same; also in (7) and (8).

If P is a point of the curve which lies in the second quadrant, $\tan\tau$ is negative, and τ is an angle of the second or fourth quadrant. If we choose to take τ as an angle of the second quadrant, formulas (8) become

$$\sin\tau = -\frac{2x}{\sqrt{1+4x^2}}, \qquad \cos\tau = -\frac{1}{\sqrt{1+4x^2}}.$$

We may, however, equally well take τ as an angle of the fourth quadrant. Then

$$\sin\tau = \frac{2x}{\sqrt{1+4x^2}}, \qquad \cos\tau = \frac{1}{\sqrt{1+4x^2}}.$$

In each case, one of the numbers, $\sin\tau$ and $\cos\tau$, is positive, the other, negative.

Polar Coordinates. Similar considerations in the case of the curve

$$r = f(\theta)$$

lead to the following formulas; cf. Fig. 45:

$$\overline{PP'}^2 = P'M^2 + MP^2.$$

Hence $$\lim_{\Delta\theta \doteq 0}\left(\frac{\overline{PP'}}{\Delta\theta}\right)^2 = \lim_{\Delta\theta \doteq 0}\left(\frac{P'M}{\Delta\theta}\right)^2 + \lim_{\Delta\theta \doteq 0}\left(\frac{MP}{\Delta\theta}\right)^2.$$

Now, the chord $\overline{PP'}$ and the arc $\overset{\frown}{PP'} = \Delta s$ are equivalent infinitesimals. Moreover, $P'M$ and Δr are equivalent; and MP and $r_0\Delta\theta$ are equivalent. Hence

$$(D_\theta s)^2 = (D_\theta r)^2 + r_0^2.$$

Dropping the subscript and writing the derivatives in terms of differentials we have, then:

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2, \tag{9}$$

or

$$ds^2 = dr^2 + r^2 d\theta^2. \tag{10}$$

Furthermore,

$$\text{(11)} \qquad \sin\psi = \frac{r\,d\theta}{ds}, \qquad \cos\psi = \frac{dr}{ds},$$

the tangent PT being drawn in the direction of the increasing s, and ψ being taken as the angle from the radius vector produced to the positive tangent.

10. Rates and Velocities. The principles of velocities and rates were treated in Chapter III, § 8. We are now in a position to deal with a wider range of problems.

We note the following formulas. Let a point P describe the curve

$$y = f(x).$$

Let s denote the length of the arc, measured from an arbitrary point in an arbitrary sense, and let τ be the angle from the positive direction of the axis of x to the tangent at P drawn in the direction of the increasing arc. Then the components of the velocity ($v = ds/dt$) of P along the axes are, respectively:

$$\text{(1)} \qquad \frac{dx}{dt} = v\cos\tau, \qquad \frac{dy}{dt} = v\sin\tau.$$

Let a point P describe the curve

$$\text{(2)} \qquad r = F(\theta).$$

Let s denote the length of the arc, measured from an arbitrary point in an arbitrary sense; and let ψ be the angle from the radius vector, produced beyond P, to the tangent at P drawn in the direction of the increasing arc. Then the components of the velocity ($v = ds/dt$) of P along the radius vector produced and perpendicular to the same (the sense of the increasing θ being taken as positive for the latter) are respectively:

$$\text{(3)} \qquad \frac{dr}{dt} = v\cos\psi, \qquad r\frac{d\theta}{dt} = v\sin\psi.$$

Example 1. A railroad train is running at the rate of 30 miles an hour along a curve in the form of a parabola:

$$y^2 = 1000\,x,$$

the axis of the parabola being east and west, and the foot being taken as the unit of length. The sun is just rising in the east. Find how fast the shadow of the locomotive is moving along the wall of the station, which is north and south, when the distance of the shadow from the axis of the parabola is 300 ft.

The first thing to do is to draw a suitable figure, introduce suitable variables, and set down all the data not already put into evidence by the figure. Thus in the present case we have, in addition to the accompanying figure, the further data: (*a*) the velocity of the train; this must be expressed in *feet per second*, since we wish to retain the foot as the unit of length for the equation of the curve. Now, 30 miles an hour is equivalent to 44 feet a second. On the other hand, another expression for the velocity is ds/dt. Hence we have, on equating these two values,

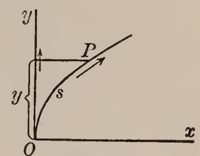

FIG. 47

$$\frac{ds}{dt} = 44.$$

(*b*) We must set down explicitly at this point the equation of the curve,

$$y^2 = 1000\,x.$$

To sum up, then, we first draw the figure and then write down the supplementary data:

Given a) $\dfrac{ds}{dt} = 44,$

and b) $y^2 = 1000\,x.$

The second thing to do is to make clear what the problem is. In the present case it can be epitomized as follows:

To find $\left(\dfrac{dy}{dt}\right)_{y=300}.$

We are now ready to consider what methods are at our disposal for solving the problem. We observe that ds occurs in

the data. Obviously, then, we must make use of the one general theorem we know which gives an expression for ds when the equation of the curve comes to us in Cartesian coordinates, — namely, the theorem:

$$ds^2 = dx^2 + dy^2.$$

Since dx occurs neither in the data nor in the conclusion, we wish to eliminate it. This can be done by means of the equation of the path b). Differentiating b) we have:

$$2\,y\,dy = 1000\,dx.$$

Hence $$dx = \frac{y\,dy}{500}.$$

Consequently $$ds^2 = \frac{y^2 dy^2}{500^2} + dy^2$$

and $$ds = \sqrt{\frac{y^2}{500^2} + 1}\,dy.$$

The next step is obvious; divide through by dt:

$$\frac{ds}{dt} = \sqrt{\frac{y^2}{500^2} + 1}\,\frac{dy}{dt}.$$

In this last equation, replace ds/dt by its known value from a), and we now have an equation for determining dy/dt:

$$\frac{dy}{dt} = \frac{44}{\sqrt{\frac{y^2}{500^2} + 1}}.$$

Finally, bring into action the particular value of y with which alone the proposed equation is concerned, namely, $y = 300$:

$$\left(\frac{dy}{dt}\right)_{y=300} = \frac{44}{\sqrt{.6^2 + 1}} = \frac{44}{\sqrt{1.36}} = 37.73,$$

or, the rate at which the shadow is moving along the wall of the station is 37.73 ft. a second.

Angular Velocity. By the *angular velocity*, ω, with which a line is turning in a given plane is meant the rate at which the angle, ϕ, made by the rotating line with a fixed line, is increasing:

$$\omega = \frac{d\phi}{dt}.$$

Example 2. A point is describing the cardioid

$$r = a(1 - \cos\theta)$$

at the rate of c ft. a second. Find the rate at which the radius vector drawn to the point is turning when $\theta = \pi/2$.

The formulation of this problem is as follows:

Given $\quad a)\quad \dfrac{ds}{dt} = c$

and $\quad b)\quad r = a(1 - \cos\theta).$*

To find $\quad \left(\dfrac{d\theta}{dt}\right)_{\theta=\frac{\pi}{2}}.$

Since, from § 9 (10),

$$ds^2 = dr^2 + r^2\,d\theta^2,$$

and from b),

$$dr = a\sin\theta\,d\theta,$$

it follows that

$$\begin{aligned} ds^2 &= a^2\sin^2\theta\,d\theta^2 + a^2(1-\cos\theta)^2\,d\theta^2 \\ &= a^2\,d\theta^2[\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta] \\ &= 2\,a^2 d\theta^2\cdot(1-\cos\theta) = 4\,a^2\sin^2\frac{\theta}{2}\,d\theta^2. \end{aligned}$$

Hence, s being measured from the cusp,

$$ds = 2\,a\sin\frac{\theta}{2}\,d\theta,$$

and

$$\frac{ds}{dt} = 2\,a\sin\frac{\theta}{2}\frac{d\theta}{dt}.$$

* The student should make a free-hand drawing of the curve.

Consequently, by the aid of a)

$$\frac{d\theta}{dt} = \frac{c}{2\,a\sin\frac{\theta}{2}},$$

and thus, finally
$$\left(\frac{d\theta}{dt}\right)_{\theta=\frac{\pi}{2}} = \frac{c}{a\sqrt{2}}.$$

EXERCISES

1. A point describes a circle of radius 200 ft. at the rate of 20 ft. a second. How fast is its projection on a fixed diameter travelling when the distance of the point from the diameter is 100 ft.? *Ans.* 10 ft. a second.

2. A flywheel 15 ft. in diameter is making 3 revolutions a second. The sun casts horizontal rays which lie in or are parallel to the plane of the flywheel. A small protuberance on the rim of the wheel throws a shadow on a vertical wall. How fast is the shadow moving when it is 4 ft. above the level of the axle?

3. A revolving light sends out a bundle of rays that are approximately parallel, its distance from the shore, which is a straight beach, being half a mile, and it makes one revolution in a minute. Find how fast the light is travelling along the beach when at the distance of a quarter of a mile from the nearest point of the beach.

4. A point moves along the curve $r = 1/\theta$ at the rate of 6 ft. a second. How fast is the radius vector turning when $\theta = 2\pi$?

5. In the example of the ladder, Chap. III, § 8, Ex. 5, find how fast the ladder is rotating at the instant in question.

6. At what rate is the direction of the second ship from the first changing at the instant in question, in Ex. 2 of Chap. III, § 8?

7. How fast is the direction of the man from the lamp-post changing in Ex. 12 of Chap. III, § 8?

8. The sun is just setting as a baseball is thrown vertically upward so that its shadow mounts to the highest point of the dome of an observatory. The dome is 50 ft. in diameter. Find how fast the shadow of the ball is moving along the dome one second after it begins to fall, and also how fast it is moving just after it begins to fall.

9. Let AB, Fig. 48, represent the rod that connects the piston of a stationary engine with the fly-wheel. If u denotes the velocity of A in its rectilinear path, and v that of B in its circular path, show that

$$u = (\sin\theta + \cos\theta \tan\phi)v.$$

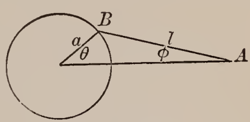

FIG. 48

10. Find the velocity of the piston of a locomotive when the speed of the axle of the drivers is given.

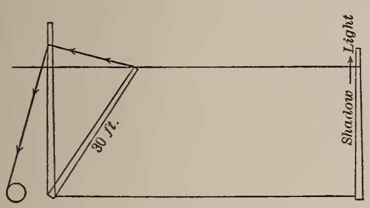

FIG. 49

11. A drawbridge 30 ft. long is being slowly raised by chains passing over a windlass and being drawn in at the rate of 8 ft. a minute. A distant electric light sends out horizontal rays and the bridge thus casts a shadow on a vertical wall, consisting of the other half of the bridge, which has been already raised. Find how fast the shadow is creeping up the wall when half the chain has been drawn in.

12. A man walks across the floor of a semicircular rotunda 100 ft. in diameter, his speed being 4 ft. a second, and his path the radius perpendicular to the diameter joining the extremities of the semicircle. There is a light at one of the latter points. Find how fast the man's shadow is moving along the wall of the rotunda when he is halfway across.

13. A man in a train that is running at full speed looks out of the window in a direction perpendicular to the track. If he fixes his attention successively for short intervals of time on objects at different distances from the train, show that the rate at which he has to turn his eyes to follow a given object is inversely proportional to its distance from him.

14. Water is flowing out of a vessel of the form of an inverted cone, whose semi-vertical angle is 30°, at the rate of a quart in 2 minutes, the opening being at the vertex. How fast is the level of the water falling when there are 4 qt. of water still in?

15. Suppose that the locomotive of the first of the Examples worked in the text is approaching the station at night at the rate of 20 miles an hour, its headlight sending out a bundle of parallel rays. How fast will the spot of light be moving along the wall of the station when the distance of the headlight from the vertex A of the parabola, measured in a straight line, is 500 ft.?

Assume that the wall is perpendicular to the axis of the parabola and distant 75 ft. from the vertex.

16. In the preceding question, how fast will the bundle of rays be rotating?

17. A point describes a circle with constant velocity. Show that the velocity with which its projection moves along a given diameter is proportional to the distance of the point from this diameter.

18. A point P describes the arc of the ellipse

$$9x^2 + 4y^2 = 36,$$

which lies in the first quadrant, at the rate of 12 ft. a second. The tangent at P cuts off a right triangle from the first quadrant. How fast is the area of this triangle changing when P passes through the extremity of the latus rectum? Is the area increasing or decreasing?

19. A point P describes the cardioid

$$r = 5(1 - \cos\theta)$$

at the rate of 12 cm. a second. The tangent at P cuts the axis of the curve in Q. How fast is Q moving when $\theta = \pi/2$?

20. The sun is just setting in the west as a horse is running around an elliptical track at the rate of m miles an hour. The axis of the ellipse lies in the meridian. Find the rate at which the horse's shadow moves on a fence beyond the track and parallel to the axis.

CHAPTER VI

LOGARITHMS AND EXPONENTIALS

1. Logarithms. The logarithms with which the student is familiar are those which are ordinarily used for computation. The base is 10, and the definition of $\log_{10} x$ is as follows:

$$y = \log_{10} x \qquad \text{if} \qquad 10^y = x.$$

These are called *denary*, or *Briggs's*, or *common* logarithms.

More generally, any positive number, a, except unity, can be taken as the base, the definition of $\log_a x$ then being:

(1) $$y = \log_a x \qquad \text{if} \qquad a^y = x.$$

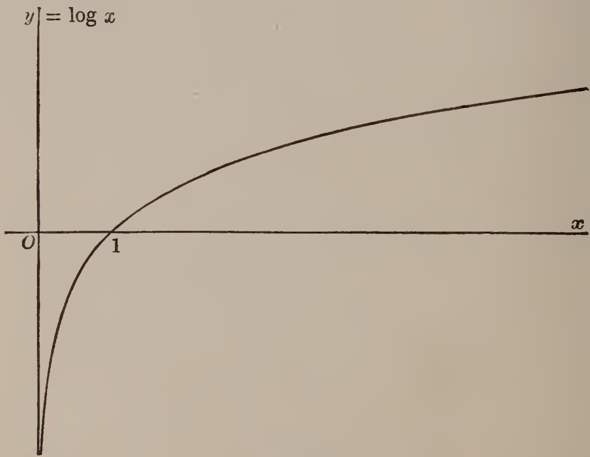

FIG. 50

The accompanying figure represents in character the graph of the function $\log_a x$ for any $a > 1$. It is drawn to scale for

$a = 2.71828$. The reason for this choice of a will appear shortly.

From the definition it follows at once that

$$(2) \qquad \log_a 1 = 0, \qquad \log_a a = 1.$$

Only positive numbers have logarithms. For, a^y is always positive. Hence, if x be given a negative value (or the value 0), the second equation under (1) above cannot be satisfied by any value of y.

The two leading properties of logarithms are expressed by the equations: *

$$\text{(I)} \qquad \log P + \log Q = \log (PQ)$$

$$\text{(II)} \qquad \log P^n = n \log P.$$

Here, P and Q are any two positive numbers whatever, and n is any number, positive, negative, or zero. The base, a, is arbitrary. Thus

$$\log 10 = \log 2 + \log 5$$

and

$$\log \sqrt{7} = \log 7^{\frac{1}{2}} = \tfrac{1}{2} \log 7.$$

From equation (I) it follows that

$$(3) \qquad \log \frac{1}{Q} = -\log Q$$

and

$$(4) \qquad \log \frac{P}{Q} = \log P - \log Q.$$

For, if we set $P = 1/Q$ in (I), we have

$$\log 1 = \log \frac{1}{Q} + \log Q.$$

But, by (2),

$$\log 1 = 0.$$

* The student should recall the proofs of these theorems, which he learned in the earlier study of logarithms, and make sure that he can reproduce them. Proofs of the theorems are given in the author's *Differential and Integral Calculus*, p. 76.

Hence
$$\log \frac{1}{Q} = -\log Q, \qquad \text{q.e.d.}$$

Again, write (1) in the form
$$\log (PQ') = \log P + \log Q',$$
and now set $Q' = 1/Q$. Then
$$\log \frac{P}{Q} = \log P + \log \frac{1}{Q}.$$

But
$$\log \frac{1}{Q} = -\log Q.$$

Hence
$$\log \frac{P}{Q} = \log P - \log Q, \qquad \text{q.e.d.}$$

For example,
$$\log (a+b) - \log a = \log \left(1 + \frac{b}{a}\right),$$
as we see by setting, in equation (4),
$$P = a + b, \qquad Q = a.$$

As a further example of the application of equation (II) we may cite the following:
$$\frac{\log (a+b)}{h} = \log \{(a+b)^{\frac{1}{h}}\}.$$

For, if $P = a + b$ and $n = \dfrac{1}{h}$, the left-hand side of this equation has the value $n \log P$.

A Further Property of Logarithms. When it is desired to express a logarithm given to a certain base, a, in terms of logarithms taken to a second base, b, the following relation is needed:

(III)
$$\log_a x = \frac{\log_b x}{\log_b a}.$$

The proof of (III) is as follows. Let
$$y = \log_a x, \qquad a^y = x.$$

Take the logarithm of each side of this equation to the base b:

(5) $$\log_b a^y = \log_b x.$$

But the left-hand side can be transformed by (II), if in (II) we take b as the base, thus having

$$\log_b P^n = n \log_b P.$$

Here, let

$$P = a, \qquad n = y.$$

Then $$\log_b a^y = y \log_b a,$$

and (5) now becomes:

$$y \log_b a = \log_b x.$$

Hence

$$y = \frac{\log_b x}{\log_b a}, \qquad \text{or} \qquad \log_a x = \frac{\log_b x}{\log_b a}, \qquad \text{q.e.d.}$$

Example. Let $b = 10$ and let $a = 2.718$. To find $\log_a 2$. From (III),

$$\log_a 2 = \frac{\log_{10} 2}{\log_{10} 2.718} = \frac{.3010}{.4343} = .6932.$$

Two Identities. Just as, for example,

$$\sqrt[3]{x^3} = x \qquad \text{and} \qquad (\sqrt[3]{x})^3 = x,$$

no matter what value x may have, so we can state two identities for logarithms and exponentials. In the second equation (1), replace y by its value from the first equation. Thus the equation

(6) $$a^{\log_a x} = x$$

is seen to hold for all positive values of x.

Secondly, replace x in the first equation (1) by its value from the second equation:

$$y = \log_a a^y.$$

We can equally well write x instead of y, understanding now by x any number whatever, and we have, then, the

identity

$$\log_a a^x = x. \tag{7}$$

This equation holds for all values of x, positive, negative, or zero.

EXERCISES

1. Show that
$$\log_{10} .8950 = -.0482.$$

2. Find $\log_{10} .09420$. *Ans.* -1.0259.

3. Compute $2.718^{.5642}$. *Ans.* 1.758.

4. Compute $2.718^{-.8710}$. *Ans.* 0.4186.

5. Compute π^π.

6. Compute $\sqrt{2}^{\sqrt{3}}$.

7. Show that
$$\log \tan \theta = \log \sin \theta - \log \cos \theta, \qquad 0 < \theta < \frac{\pi}{2}.$$

8. Show that
$$\log \sin \theta + \log \cos \theta = \log \frac{\sin 2\theta}{2}, \qquad 0 < \theta < \frac{\pi}{2}.$$

9. Show that
$$\log \frac{1 - \cos \theta}{2} = 2 \log \sin \frac{\theta}{2}, \qquad 0 < \theta < 2\pi.$$

10. If (x, y) are the Cartesian coordinates of a point distinct from the origin, and (r, θ) the polar coordinates of the same point, show that
$$\log r = \tfrac{1}{2} \log (x^2 + y^2).$$

11. Prove that
$$\log (a^2 - b^2) = \log (a + b) + \log (a - b),$$
provided $a + b$ and $a - b$ are both positive quantities.

12. Simplify the expression
$$\log (1 + x^6) - \log (1 + x^2).$$

13. Show that
$$\sqrt{(e^x - e^{-x})^2 + 4} = e^x + e^{-x},$$
where e has the value 2.7182.

14. Simplify the expression

$$\sqrt{\left(\frac{a^x - a^{-x}}{2}\right)^2 + 1}, \qquad 0 < a.$$

15. Show that

$$\frac{1}{t} \log (1 + t) = \log (1 + t)^{\frac{1}{t}}.$$

2. Differentiation of Logarithms. In order to differentiate the function

$$y = \log_a x,$$

it is necessary to go back to the definition of a derivative, Chap. II, § 1, and carry through the process step by step.

Give to x an arbitrary positive value, x_0, and compute the corresponding value, y_0, of the function:

(1) $$y_0 = \log_a x_0.$$

Next, give to x an increment Δx (subject merely to the restriction that $x_0 + \Delta x$ is positive and $\Delta x \neq 0$) and compute the new value, $y_0 + \Delta y$, of the function:

(2) $$y_0 + \Delta y = \log_a (x_0 + \Delta x).$$

From (1) and (2) it follows that

$$\frac{\Delta y}{\Delta x} = \frac{\log_a (x_0 + \Delta x) - \log_a x_0}{\Delta x}.$$

It is at this point that the specific properties of the logarithmic function come into play for the purpose of transforming the last expression. By § 1, (4),

$$\log_a (x_0 + \Delta x) - \log_a x_0 = \log_a \left(1 + \frac{\Delta x}{x_0}\right),$$

and hence

(3) $$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x_0}\right).$$

We next replace the variable Δx by a new variable t as follows:

$$t = \frac{\Delta x}{x_0} \qquad \text{or} \qquad \Delta x = x_0 t.$$

Thus (3) takes on the form

$$\frac{\Delta y}{\Delta x} = \frac{1}{x_0 t} \log_a (1 + t) = \frac{1}{x_0}\left[\frac{1}{t} \log_a (1 + t)\right].$$

From (II), § 1, the bracket is seen to have the value

$$\log_a (1 + t)^{\frac{1}{t}},$$

and hence

$$(4) \qquad \frac{\Delta y}{\Delta x} = \frac{1}{x_0} \log_a (1 + t)^{\frac{1}{t}}.$$

As Δx approaches 0 as its limit, t also approaches 0, and so

$$(5) \qquad \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \frac{1}{x_0} \lim_{t \doteq 0} \log_a (1 + t)^{\frac{1}{t}}.$$

Now, the variable $(1 + t)^{\frac{1}{t}}$ approaches a limit when t approaches 0, and this limit is the number which is represented in mathematics by the letter e; cf. § 3. Its value to five places of decimals is

$$e = 2.71828 \cdots;$$

cf. § 3. Moreover, $\log x$ is a continuous function of x, as is shown in a detailed study of this function.* Hence

$$\lim_{t \doteq 0} \log_a (1 + t)^{\frac{1}{t}} = \log_a \left\{ \lim_{t \doteq 0} (1 + t)^{\frac{1}{t}} \right\} = \log_a e.$$

On substituting this value in the right-hand side of (5) we have:

$$D_x y = \frac{\log_a e}{x_0},$$

* Such a treatment is too advanced to be pursued with profit at this stage. Cf. the author's *Differential and Integral Calculus*, Appendix, p. 417.

or, on dropping the subscript:

(6) $$D_x \log_a x = \frac{\log_a e}{x}.$$

Thus if the usual base, $a = 10$, be taken, the formula becomes:

(7) $$D_x \log_{10} x = \frac{.4343 \cdots}{x}.$$

Discussion of the Result. We have met a similar situation before, in the differentiation of the sine. There, if angles be measured in degrees, the fundamental formula reads:

$$D_x \sin x = \frac{\pi}{180} \cos x.$$

In order to get rid of this inconvenient multiplier, we changed the unit of angle from the degree to the radian, and then the formula became:

$$D_x \sin x = \cos x.$$

In the present case, it is possible to do a similar thing. The base, a, is wholly in our control, to choose as we like. Now, for any base, the logarithm of the base is unity, § 1, (2):

$$\log_a a = 1.$$

If, then, we choose as our base the number e:

$$a = e = 2.71828 \cdots$$

the multiplier becomes

(8) $$\log_a e = \log_e e = 1.$$

For this reason, e is taken as the base of the logarithms used in the Calculus.* These are called NATURAL *logarithms.* They are also called *hyperbolic*, or *Naperian* logarithms, — the latter name after Napier, the inventor of logarithms. But

* The notation e for this number is due to Euler, 1728.

Napier * was the very man who introduced denary logarithms into mathematics, and so the use of his name in connection with natural logarithms is misleading.

Since natural logarithms are always meant in the formulas of the calculus, unless the contrary is explicitly stated, it is customary to drop the index e from the notation $\log_e x$ and to write

$$y = \log x, \qquad \text{if} \qquad e^y = x. \tag{9}$$

The identities (6) and (7) of § 1 now take on the form:

$$e^{\log x} = x, \tag{10}$$

$$\log e^x = x. \tag{11}$$

The formula of differentiation becomes:

$$D_x \log x = \frac{1}{x}. \tag{12}$$

In differential form it reads:

$$\frac{d}{dx} \log x = \frac{1}{x}, \tag{13}$$

$$d \log x = \frac{dx}{x}. \tag{14}$$

Example. Differentiate the function

$$u = \log \sin x.$$

Let
$$y = \sin x.$$

Then
$$u = \log y,$$

$$du = d \log y = \frac{dy}{y}, \qquad dy = \cos x\, dx,$$

and

$$du = \frac{\cos x\, dx}{\sin x} = \cot x\, dx.$$

* Napier was a Scotchman, and his discovery was published in 1614.

Hence $$d \log \sin x = \cot x \, dx,$$

or $$\frac{d}{dx} \log \sin x = \cot x.$$

EXERCISES

Differentiate the following functions.

1. $u = \log \cos x.$ $\qquad \dfrac{du}{dx} = -\tan x.$

2. $u = \log \tan x.$ $\qquad \dfrac{du}{dx} = \cot x + \tan x.$

3. $u = \log \cot x.$ $\qquad \dfrac{du}{dx} = \dfrac{-2}{\sin 2x}\cdot$

4. $u = \log \sec x.$ $\qquad$ 5. $u = \log \csc x.$

6. $u = \log \dfrac{x}{1-x}\cdot$ $\qquad \dfrac{du}{dx} = \dfrac{1}{x} + \dfrac{1}{1-x}\cdot$

7. $u = \log \dfrac{a+x}{a-x}\cdot$ $\qquad \dfrac{du}{dx} = \dfrac{2a}{a^2 - x^2}\cdot$

8. $u = \log \sqrt{a^2 + x^2}.$ $\qquad \dfrac{du}{dx} = \dfrac{x}{a^2 + x^2}\cdot$

9. $u = \log (1 - \cos x).$ $\qquad \dfrac{du}{dx} = \cot \dfrac{x}{2}\cdot$

10. $u = \log (1 + \cos x).$ $\qquad \dfrac{du}{dx} = -\tan \dfrac{x}{2}\cdot$

3. The Limit $\lim\limits_{t \doteq 0} (1+t)^{\frac{1}{t}}$. Since this limit is fundamental in the differentiation of the logarithm, a detailed discussion of it is essential to completeness. Let us set

$$s = (1+t)^{\frac{1}{t}} \tag{1}$$

and compute the value of s for values of t near 0. Suppose $t = .1$. Then

$$s = (1.1)^{10},$$

and this number is found by the usual processes with logarithms to be 2.59.

Further pairs of corresponding values (t, s) are found in a similar manner. In particular, the student can verify the correctness of the following table of values:*

t	-0.1	$-.01$	$-.001$	$\ldots$	$+.001$	$+.01$	$+0.1$
s	2.87	2.73	2.72	$\ldots$	2.72	2.70	2.59

The foregoing table indicates strongly that, when t approaches the limit 0 from either side, the variable s is approaching a limit whose value, to three significant figures, is 2.72. This is in fact the case.† The exact value of the limit is denoted by the letter e:

$$\lim_{t \doteq 0} (1+t)^{\frac{1}{t}} = e = 2.71828 \cdots. \tag{2}$$

4. The Compound Interest Law. The limit (2) of § 3 presents itself in a variety of problems, typical for which is that of finding how much interest a given sum of money would bear if the interest were compounded continuously, so that there is no loss whatever. For example, \$1000, put at interest at 6%, amounts in a year to \$1060, if the interest is not compounded at all. If it is compounded every six months, we have

$$\$\,1000\left(1 + \frac{.06}{2}\right)$$

as the amount at the end of the first six months, and this must be multiplied by $\left(1 + \frac{.06}{2}\right)$ to yield the amount at the end of the second six months, the final amount thus being

$$\$\,1000\left(1 + \frac{.06}{2}\right)^2.$$

* To compute the middle entries in this table a six-place table of logarithms is needed.

† For a rigorous proof cf. the author's *Differential and Integral Calculus*, p. 79.

It is readily seen that if the interest is compounded n times in a year, the principal and interest at the end of the year will amount to

$$1000\left(1+\frac{.06}{n}\right)^n$$

dollars, and we wish to find the limit of this expression when $n=\infty$. To do so, write it in the form:

$$1000\left[\left(1+\frac{.06}{n}\right)^{\frac{n}{.06}}\right]^{.06}$$

and set $t=\dfrac{.06}{n}$. The bracket thus becomes

$$(1+t)^{\frac{1}{t}},$$

and its limit is e. Hence the desired result is

$$1000\,e^{.06}=1061.84.*$$

EXERCISE

If \$1000 is put at interest at 4 %, compare the amounts of principal and interest at the end of 10 years, (*a*) when the interest is compounded semiannually, and (*b*) when it is compounded continuously. *Ans.* A difference of \$5.88.

5. Differentiation of e^x. Before beginning this paragraph the student will turn to Chap. VIII and study carefully § 1.

Since

(1) $\qquad y=e^x \qquad$ and $\qquad x=\log y$

are equivalent equations, the former function can be differentiated by taking the differential of each side of the latter equation:

$$dx=d\log y=\frac{dy}{y}.$$

* The actual computation here is expeditiously done by means of series; see the chapter on Taylor's Theorem.

Hence
$$\frac{dy}{dx} = y,$$
or

(2) $$\frac{de^x}{dx} = e^x,$$

(3) $$de^x = e^x\,dx.$$

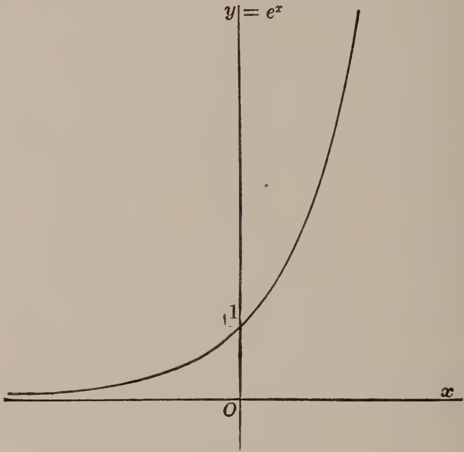

FIG. 51

The function

(4) $$y = a^x$$

could be differentiated in a similar manner. It is, however, simpler to take the logarithm of each side of (4) and then differentiate the new equation:

$$\log y = \log a^x = x \log a,$$

$$d \log y = \frac{dy}{y} = dx \log a.$$

(5) $$d a^x = a^x \log a\, dx.$$

Differentiation of x^n. It is now possible to complete the differentiation of this function for the case that n is irrational.

Since by § 2, (10), $x = e^{\log x}$,

it follows that $x^n = e^{n \log x}$,

and hence

$$d\,x^n = d\,e^{n \log x}$$
$$= e^{n \log x}\, d\,(n \log x)$$
$$= e^{n \log x}\,\frac{n\,dx}{x}$$
$$= x^n\,\frac{n\,dx}{x}.$$

Thus finally,

$$d\,x^n = nx^{n-1}\,dx, \tag{6}$$

no matter what value n may have, provided merely that n is a constant.

Differentiation of $f(x)^{\phi(x)}$. Let it be required, for example, to differentiate the function

$$y = x^x.$$

Here, both base and exponent are variable. Begin by taking the logarithm of each side of the equation:

$$\log y = \log x^x = x \log x.$$

Hence

$$d \log y = d(x \log x),$$

or

$$\frac{dy}{y} = (1 + \log x)\,dx,$$

and so, finally,

$$dy = y\,(1 + \log x)\,dx$$

or

$$d\,x^x = x^x\,(1 + \log x)\,dx.$$

The general case, $y = f(x)^{(\phi x)}$,

can be treated in a similar manner.

6. Graph of the Function x^n. For positive values of n the curves

$$y = x^n$$

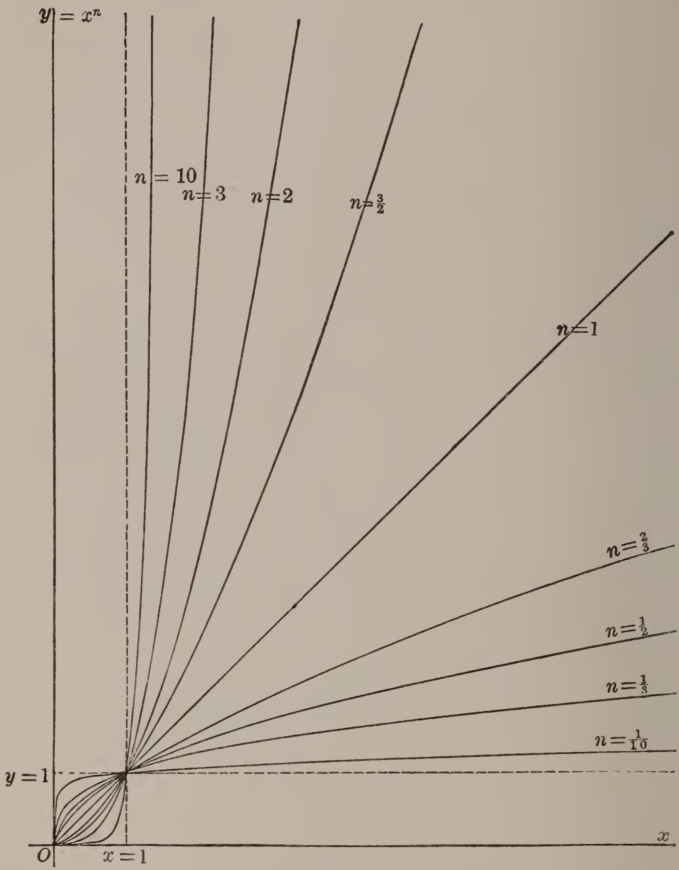

FIG. 52

lie as indicated in the figure. When $n = 1$, we have the ray from the origin, which bisects the angle between the positive axes of x and y.

When $n > 1$, the curve is always concave upward; when $n < 1$, it is concave downward.

All the curves start at the origin and pass through the point (1, 1).

For values of $x > 1$, the larger n, the higher the curve lies. For values of $x < 1$, the reverse is the case.

Let x have any fixed value greater than unity: $x = x' > 1$. Consider the ordinate

$$y = x'^n.$$

As n increases, x'^n increases continuously. This property is the basis of the property of logarithms included in the word *continuous*.

For proofs of the foregoing statements cf. the author's *Differential and Integral Calculus*, p. 27 and Appendix, p. 417.

7. The Formulas of Differentiation to Date. The student will now bring his card of formulas up to date by supplementing it so that it will read as follows:

GENERAL FORMULAS OF DIFFERENTIATION

I. $$d\,cu = c\,du.$$

II. $$d(u + v) = du + dv.$$

III. $$d(uv) = u\,dv + v\,du.$$

IV. $$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

SPECIAL FORMULAS OF DIFFERENTIATION

1) $$dc = 0.$$

2) $$d\,x^n = nx^{n-1}\,dx.$$

3) $$d\sin x = \cos x\;dx.$$

4) $$d\cos x = -\sin x\;dx.$$

5) $$d\tan x = \sec^2 x\;dx.$$

6) $$d \cot x = -\csc^2 x \, dx.$$

7) $$d \log x = \frac{dx}{x}.$$

8) $$d e^x = e^x \, dx.$$

9) $$d a^x = a^x \log a \, dx.$$

To obtain facility in the use of the new results it is desirable that the student work a good number of simple exercises.

Example 1. To differentiate the function

$$u = e^{ax}.$$

Let $$y = ax.$$

Then $$u = e^y,$$

$$du = d e^y = e^y \, dy = e^{ax}(a \, dx).$$

Hence

$$d e^{ax} = a e^{ax} \, dx \qquad \text{or} \qquad \frac{d}{dx} e^{ax} = a e^{ax}.$$

Example 2. If

$$u = A \cos(nt + \gamma),$$

show that *

$$\frac{d^2u}{dt^2} + n^2 u = 0.$$

To do this, compute first $\frac{du}{dt}$. The computation is readily effected by taking the differential of each side of the given equation:

$$\begin{aligned} du &= A \, d \cos(nt + \gamma) \\ &= A[-\sin(nt + \gamma) \, d(nt + \gamma)] \\ &= -An \sin(nt + \gamma) \, dt, \end{aligned}$$

* Such an equation as the following is called a *differential equation*, and any function which, when substituted for u, satisfies the equation is called a *solution*.

$$\frac{du}{dt} = -An \sin(nt+\gamma).$$

Next, compute $\frac{d^2u}{dt^2}$. Since

$$\frac{d^2u}{dt^2} = \frac{d}{dt}\left(\frac{du}{dt}\right) = \frac{d\left(\frac{du}{dt}\right)}{dt},$$

we take the differential of each side of the equation for $\frac{du}{dt}$:

$$d\left(\frac{du}{dt}\right) = -An\, d\sin(nt+\gamma)$$

$$= -An[\cos(nt+\gamma)d(nt+\gamma)]$$

$$= -An^2\cos(nt+\gamma)\,dt.$$

Hence, on dividing through by dt, we have:

$$\frac{d^2u}{dt^2} = -An^2\cos(nt+\gamma).$$

If now we multiply the given value of u by n^2 and add the product to the value just obtained for $\frac{d^2u}{dt^2}$, the result is identically 0, *i.e.* 0 for all values of t:

$$\frac{d^2u}{dt^2} + n^2u = 0, \qquad \text{q. e. d.}$$

EXERCISES

Differentiate the following functions.

1. $u = e^{-x^2}$. $\qquad \frac{du}{dx} = -2xe^{-x^2}$.

2. $u = e^{\sin x}$. $\qquad \frac{du}{dx} = e^{\sin x}\cos x$.

3. $u = (e^x + e^{-x})^2$. $\qquad \frac{du}{dx} = 2(e^{2x} - e^{-2x})$.

4. $u = 10^x.$ $\qquad \dfrac{du}{dx} = (2.30259 \cdots)10^x.$

5. $u = x^{10}\,10^x.$ $\qquad \dfrac{du}{dx} = x^9\,10^x(10 + 2.30259\,x).$

6. $u = \log(\sec x + \tan x).$ $\qquad \dfrac{du}{dx} = \sec x.$

7. $u = x^2 \log x.$ $\qquad \dfrac{du}{dx} = x(1 + 2\log x).$

8. $u = x^3 \log(a - x).$

9. $u = e^{-x}\log(2x + 3).$

10. $u = e^{-at}\cos(nt - \gamma).$

11. $u = e^{-\kappa t}(A\cos nt + B\sin nt).$

12. $u = \dfrac{x\log x}{x+1} - \log(x+1).$ $\qquad \dfrac{du}{dx} = \dfrac{\log x}{(x+1)^2}.$

13. $u = \log(x + \sqrt{x^2 - a^2}).$ $\qquad \dfrac{du}{dx} = \dfrac{1}{\sqrt{x^2 - a^2}}.$

14. $u = \log(x + \sqrt{a^2 + x^2}).$ $\qquad \dfrac{du}{dx} = \dfrac{1}{\sqrt{a^2 + x^2}}.$

15. $u = \log(e^x + e^{-x}).$

16. $u = \dfrac{\sin x + \cos x}{e^x}.$

17. $u = \log\tan\dfrac{x}{2}.$ $\qquad \dfrac{du}{dx} = \csc x.$

18. $u = \log\tan\left(\dfrac{\theta}{2} + \dfrac{\pi}{4}\right).$ $\qquad \dfrac{du}{d\theta} = \sec\theta.$

19. $u = \cot\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right).$ $\qquad \dfrac{du}{dx} = \dfrac{1}{1 - \sin x}.$

20. $u = \tan\left(\dfrac{x}{2} - \dfrac{\pi}{4}\right).$ $\qquad \dfrac{du}{dx} = \dfrac{1}{1 + \sin x}.$

21. $u = \log\sqrt{1 + \sin\theta}.$

22. $u = \log\dfrac{\sin x}{x}.$

23. $u = \log\sqrt{1 - \cos x}.$

24. $u = \sqrt{e^x}.$

25. $u = (10^{1+t})^2.$

26. $u = \sqrt[3]{a^{-x}}.$

27. $u = \left(\dfrac{a}{a+b}\right)^x.$

28. $u = \sqrt{10^x}.$

29. $u = x^{\sin x}$. $\quad \dfrac{du}{dx} = x^{\sin x - 1}(\sin x + x \cos x \log x)$.

30. $u = (\sin x)^{\cos x}$. $\quad \dfrac{du}{dx} = (\sin x)^{\cos x - 1}(\cos^2 x - \sin^2 x \log \sin x)$.

31. $u = x^{\frac{1}{x}}$. **32.** $u = (\cos x)^{\sin x}$. **33.** $u = (\tan x)^x$.

34. $u = (\log x^2)^x$. **35.** $u = (1 + a)^{\frac{x}{a}}$. **36.** $u = (x^2)^{2x}$.

37. If $u = A \cos nt + B \sin nt$, show that

$$\frac{d^2u}{dt^2} + n^2 u = 0.$$

38. If $u = Ce^{-\kappa t} \cos(\sqrt{n^2 - \kappa^2}\, t + \gamma)$, show that

$$\frac{d^2u}{dt^2} + 2\kappa \frac{du}{dt} + n^2 u = 0.$$

CHAPTER VII

APPLICATIONS

1. The Problem of Numerical Computation. It often happens in practice that we wish to solve a numerical equation in one unknown quantity, or a pair of simultaneous equations in two unknowns, to which the standard methods with which we are familiar do not apply; for example,

$$\cos x = x,$$

or

$$\begin{cases} 2\cot\theta + 2 = \cot\phi, \\ 2\cos\theta + \cos\phi = 2. \end{cases}$$

Such equations usually come to us from physical problems, and the solution is required only to a limited degree of accuracy, — say, to two, three, or possibly four significant figures. Any method, therefore, which yields an approximate solution correct to the prescribed degree of accuracy furnishes a solution of the problem.

In particular, the problem of the determination of the error in the result due to errors in the observations comes under this head.

2. Solution of Equations. Known Graphs.

Example 1. Let it be required to solve the equation

1) $$\cos x = x.$$

We can evidently replace this problem by the following: To find the abscissa of the point of intersection of the curves

2) $$y = \cos x, \qquad y = x.$$

The first of these curves we have plotted accurately to scale. The second is the right line through the origin, which bisects the angle between the positive coordinate axes. It is, therefore, sufficient to lay down a ruler on the graph of the former curve, so that its edge lies along the right line in question, and observe where this line cuts the curve. The result lies between

$$x = .7 \qquad \text{and} \qquad x = .8,$$

and may fairly be taken as $x = .75$. It is understood, as usual in approximate values, that the last figure tabulated does not claim complete accuracy; but we are entitled to a somewhat better result than would be given by the first figure alone.

Example 2. To solve the equation

3) $$x^3 + 2x - 2 = 0.$$

Suppose we have plotted the curve

4) $$y = x^3$$

accurately from a table of cubes. Then the problem can conveniently be formulated as follows:

To find the abscissa of the point of intersection of the curves

5) $$y = x^3 \qquad \text{and} \qquad y = 2 - 2x.$$

The details are left to the student.

Example 3. To find the positive root of the equation

6) $$e^{-\frac{1}{2}x} + 2.92x = 2.14.$$

Here, we can connect up with the graph of the function e^x by making a simple transformation. Let

7) $$x' = -\tfrac{1}{2}x; \qquad x = -2x'.$$

The equation then becomes

8) $$e^{x'} - 5.84x' = 2.14,$$

and we seek to determine the abscissa of that point of intersection of the curves (for simplicity, we drop the accent)

$$9) \qquad y = e^x \qquad \text{and} \qquad y = 5.84x + 2.14$$

which lies to the left of the origin. The second place of decimals in the coefficients is not to be taken too seriously; we make as accurate a drawing as the graph and a well-sharpened pencil permit. Having thus determined the negative x' from the graphs of 9), we find the desired positive x by substituting this value in equations 7). The execution of the details is left to the student.

Example 4. Solve the equation

$$e^x = \tan x, \qquad 0 < x < \frac{\pi}{2}.$$

If one of the curves

$$y = e^x \qquad \text{or} \qquad y = \tan x$$

were plotted on transparent paper, or celluloid, it could be laid down on the other with the axes coinciding and the intersection read off. The same result can be attained by holding the actual graphs up in front of a bright light.

In cases as simple as this, however, free-hand graphs will often yield a good first approximation, and further approximations can be secured by the numerical methods of the later paragraphs.

EXERCISES *

1. Solve the equation

$$\cos x = 2x.$$

2. Find the root of the equation

$$3 \sin x = 2x$$

which lies between 0 and π.

* In solving these exercises only so great accuracy is expected as can be attained from well-drawn graphs of the standard curves. It will be shown in later paragraphs how the solutions can be improved analytically and carried to any desired degree of accuracy.

3. Solve: $x + \tan x = 1, \qquad 0 < x < \frac{\pi}{2}.$

4. Solve: $3 \cos x - 5x = 6, \qquad -\frac{\pi}{2} < x < 0.$

5. Find the root of the equation

$$\log x^2 + 2 = x$$

which lies between 0 and 1.

6. Solve: $\sin 2x = x.$

7. Find all the roots of the equation

$$12x^3 + 4x + 3 = 0.$$

8. The same for $6x^3 - 5x - 1 = 0.$

9. The same for $x^3 - x - 1 = 0.$

Solve the following equations:

10. $\cos^3 \theta + .47 \cos \theta - 1.23 = 0, \qquad 0 < \theta < 90°.$

11. $\sin x = \sqrt{1 - x^2}.$ 12. $x^2 + \cos^2 x = 4.$

13. Show that the equation

$$\tan x = x$$

has an infinite number of roots. These can be written in the form

$$x_n = n\pi + \epsilon_n,$$

where ϵ_n is numerically small when n is numerically large.

14. Find the largest value of P for which the equation

$$\cos x + Px = 1$$

admits a solution in the interval $0 < x < \pi$.

15. Find the point of the parabola

$$2y = x^2$$

which is nearest to the point (2, 0).

16. Find the radius of the circle whose center is at $(0, 2)$ and which is tangent to the parabola

$$y^2 = x.$$

3. Interpolation. Consider the equation

$$f(x) = 0. \tag{1}$$

Suppose a root has been located with some degree of accuracy. More precisely, suppose that

$$f(x_1) \qquad \text{and} \qquad f(x_2)$$

are of opposite signs. If the function $f(x)$ is continuous in the interval $x_1 \leqq x \leqq x_2$ and if its derivative is always positive (or always negative) in this interval, then the function is always increasing (or always decreasing) and so must have just one root between x_1 and x_2.

The root can be found approximately as follows. Consider the graph of the function

$$y = f(x). \tag{2}$$

Let
$$y_1 = f(x_1), \qquad y_2 = f(x_2),$$

and draw the chord through the points (x_1, y_1) and (x_2, y_2). The point in which this chord cuts the axis of x will obviously yield a further approximation to the root sought. Denote this last value by X.

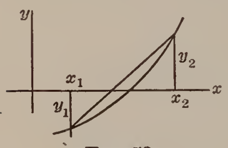

FIG. 53

The equation of the chord is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}. \tag{3}$$

On setting $y = 0$ and solving for x, we have, as the value of X, the following:

$$X = x_1 - \frac{x_2 - x_1}{y_2 - y_1} y_1, \tag{4}$$

or

$$X = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1). \tag{5}$$

We have explained the method in detail and developed, in equations 4) and 5), the analytic formula for the determination of the new approximation, X. In practice, however, it is usually simpler to draw the straight lines of Fig. 53 accurately on a generous scale and read off from the figure the value of X.

Example. Consider equation 3) of § 2, Ex. 2:

6) $$x^3 + 2x - 2 = 0.$$

The curve in question is here

7) $$y = x^3 + 2x - 2,$$

and the graphical solution of § 2 shows that the root is about $x = .7$ or $.8$.

Let $x = x_1 = .7$; then y_1 is found to have the value

$$y_1 = - .257.$$

Next, let $x = x_2 = .8$; then $y_2 = .112$.

We have, then, to lay a secant through the points

$$(x_1, y_1) = (.7, - .257) \qquad \text{and} \qquad (x_2, y_2) = (.8, .112).$$

Its equation is given by 3)*:

$$\frac{x - .7}{.8 - .7} = \frac{y + .257}{.112 + .257}.$$

On setting $y = 0$ in this equation and solving for x, we get, cf. 4):

$$X = .7 + \frac{.0257}{.369} = .7693.$$

In order to see about how close this approximation is, compute the corresponding value of y:

$$y \,|_{x = .7693} = - .0063.$$

We get, then, about two places of decimals, $x = .77$.

* It is desirable that the student should make this determination graphically, as indicated above in the text. He should take 10 cm. to represent the interval of length .1, from $x_1 = .7$ to $x_2 = .8$.

It is possible to apply the method again, taking now

$$(x_1, y_1) = (.7693, -.0063)$$

and (x_2, y_2) as before. We leave this as an exercise to the student. He should make both the graphical determination with an enlarged scale and the analytic determination of formula 4).

The Method; Not, the Formula. The student may be tempted to use the formula 4) or 5), rather than to go back to the method by which it was derived. This would be unfortunate, for the formula is not easily remembered, whereas the method, once appreciated, can never be forgotten. If the student finds himself in a lumber camp with nothing but the ordinary tables at hand, he may solve his equation if he has once laid hold of the method. It is true that the best way is for him to treat first the literal case and deduce the formula. But this he may not be able to do if he has relied on the formula in the book.

EXERCISES

Apply the method to a good number of the problems at the end of § 2.

4. Newton's Method. Suppose again that it is a question of solving the equation

$$1) \qquad f(x) = 0,$$

and suppose we have already succeeded in finding a fairly good approximation, $x = x_1$.

Consider the graph of the function

$$2) \qquad y = f(x).$$

Compute $y_1 = f(x_1)$. To improve the approximation, draw the tangent at the point (x_1, y_1). Its equation is:

$$3) \qquad y - y_1 = \left(\frac{dy}{dx}\right)_{x=x_1} (x - x_1).$$

Evidently, this line will cut the axis of x at a point very near the point in which the curve 2) cuts this axis. If, then, we set $y=0$ in 3) and solve for x, we shall obtain a second approximation to the root of 1) which we seek. The value of this root will be

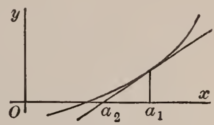

FIG. 54

$$4)\qquad X = x_1 - \frac{y_1}{\left(\dfrac{dy}{dx}\right)_{x=x_1}}.$$

Example 1. Let us apply the method to the Example studied in § 3. In order, however, to have simpler numbers to work with, take $x_1 = .77$ and compute the corresponding y_1; it is found to be: $y_1 = -.0035$.

$$(x_1, y_1) = (.77, -.0035).$$

We must next compute dy/dx from the equation

$$y = x^3 + 2x - 2;$$

$$\frac{dy}{dx} = 3x^2 + 2, \qquad \left(\frac{dy}{dx}\right)_{x=.77} = 3.779.$$

On substituting these values in 3), we have:

$$y + .0035 = 3.779(x - .77).$$

Now set $y=0$ and solve. The result is that given by 4):

$$x = .77 + \frac{.0035}{3.779} = .7709.$$

We have tabulated four figures in the result because this is about the degree of accuracy that seems likely. To test this point, compute y for the value of x which has been found:

$$y\,|_{x=.7709} = -.0001.$$

Since the slope of the graph is greater than unity, the error in x is less than one unit in the fourth place. It is easy to verify the result by computing y for the next larger four-place value of x:

$$y\,|_{x=.7710} = +.0003.$$

Thus we have a complete proof that the root lies between .7709 and .7710, and we see that it lies about one quarter of the way from the first to the second value.

Example 2. It is shown that the equation of the curve in which a chain hangs, — the *Catenary*, — is

5) $$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

where a is a constant. The length of the arc, measured from the vertex, is

6) $$s = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right).$$

Let it be required to compute the dip in a chain 32 feet long, its ends being supported at the same level, 30 feet apart.

We can determine the dip from 5) if we know a, and we can get the value of a from 6) by setting $s = 16$, $x = 15$:

$$16 = \frac{a}{2}\left(e^{\frac{15}{a}} - e^{-\frac{15}{a}}\right).$$

Let $x = \dfrac{15}{a}$. Then

$$f(x) = e^x - e^{-x} - \tfrac{32}{15}x = 0,$$

and we wish to know where the curve

7) $$y = f(x) = e^x - e^{-x} - \tfrac{32}{15}x$$

crosses the axis of x.

This curve starts from the origin and, since

$$\frac{dy}{dx} = f'(x) = e^x + e^{-x} - \tfrac{32}{15}$$

is negative for small values of x, the curve enters the fourth quadrant. Moreover,

$$\frac{d^2y}{dx^2} = e^x - e^{-x} > 0, \qquad x > 0,$$

and hence the graph is always concave upward. Finally,

$$f(1) = e - e^{-1} - 2\tfrac{2}{15} = .217 > 0,$$

and so the equation has one and only one positive root, and this root lies between 0 and 1.

It will probably be better to locate the root with somewhat greater accuracy before beginning to apply the above method. Let us compute, therefore, $f(\frac{1}{2})$. By the aid of Peirce's Tables we find:

$$f(.5) = 1.6487 - .6065 - 1.0667 = -.0245 < 0.$$

Comparing these two values of the function:

$$f(.5) = -.02, \qquad f(1) = .22,$$

and remembering that the curve is concave upward, so that the root is somewhat larger than the value obtained by direct interpolation (this value corresponding to the intersection of the chord with the axis of x) we are led to choose as our first approximation $x_1 = .6$:

$$f(.6) = 1.8221 - .5488 - 1.2800 = -.0067,$$

$$f'(.6) = 1.8221 + .5488 - 2.1333 = .2376.$$

Hence the value of the next approximation is

$$X = .6 - \frac{-.0067}{.2376} = .6 + .0282 = .628.$$

To get the next approximation we compute

$$f(.628) = 1.8739 - .5337 - 1.3397 = .0005.$$

Hence the value of the root to three significant figures is .628 with a possible error of a unit or two in the last place, and the value of a we set out to compute is, therefore, $15/.628 = 23.9$.

Remark. Newton's method, like the other methods of this chapter, has the advantage that an error in computing the new approximation will not be propagated in later computations. Such an error will in general hinder us, because we are not

likely to get so good an approximation. But the one test for the accuracy of the approximation is the accurate computation of the corresponding y, and if this is done right, we see precisely how close we are to the desired root.

The function $f(x)$ is usually simple, and it is easy to see whether the curve is concave upward or concave downward near the point where it crosses the axis. We thus have a means of improving the approximation at the same time that we simplify the new value of x. For, if the curve lies to the right of its chord, the approximation by interpolation will be too small; and if the curve lies to the right of its tangent between the point of tangency and the axis of x, the approximation given by Newton's method will also be too small.

Comparison of the Two Methods. When looked at from their geometric side the two methods appear much alike, the first seeming somewhat simpler, since it does not involve the use of derivatives. Why bother, then, with Newton's method? It is not a theoretical question, but purely one of convenience in carrying out the numerical work. It will be found that, as a rule, the first method is preferable in the early stages (usually, merely in the first stage). When, however, a fairly good approximation has been reached, the numerical work involved in Newton's method is generally shorter than that required by interpolation.

EXERCISES

Apply the method to the Exercises of § 2. When, however, the approximation given by the graphical method of § 1 is crude, the method of interpolation may be used to improve it.

5. Direct Use of the Tables.

Example 1. Let us recur to the first example studied, Ex. 1, § 2:

1) $$\cos x = x.$$

The graphical solution gave $x = .75$. Turn now to a table of natural cosines in radian measure, preferably Peirce's Tables. As we run down the table, we find the entries:

RADIANS	COS NAT
.7389	.7392
.7418	.7373

Thus x is seen to lie between .7389 and .7418. It is an excellent exercise for the student to work out the interpolation for himself before we take it up at the end of the paragraph. The answer is: $x = .7391$.

Example 2. Consider the equation

2) $$\tan x = e^x,$$

the desired root lying between 0 and $\pi/2$.

A free-hand drawing of the graphs of the functions

$$y = \tan x, \qquad y = e^x$$

shows that x lies between 1 and 1.5. So the next step is taken conveniently by opening Peirce's Tables to the Trigonometric Functions and Huntington's to the Exponentials, and writing down the two pairs of values of the functions which came nearest together:

x	$\tan x$	e^x
1.3	3.60	3.67
1.4	5.80	4.06

Thus the root is seen to lie between 1.3 and 1.4.

The general case which the above examples are intended to illustrate is the following: — To solve the equation

$$f(x) = \phi(x),$$

where $f(x)$ and $\phi(x)$ are tabulated functions, or functions readily computed.

When the solution has progressed to the point indicated by the examples, the next step can be taken by interpolation, or by Newton's method, as will now be explained.

Interpolation. When two values of the independent variable near together, x_1 and x_2, have been found such that $f(x)$ is greater than $\phi(x)$ for one of them and less than $\phi(x)$ for the other, the best approximation to take next is the one given by the abscissa of the point of intersection of the chords of the graphs of the functions,

$$y = f(x), \qquad y = \phi(x).$$

This value, X, can be found as follows.

Suppose that

$$f(x_1) < \phi(x_1) \qquad \text{and} \qquad f(x_2) > \phi(x_2).$$

Introduce the following notation:

$$\phi(x_1) - f(x_1) = \Delta_1, \qquad f(x_2) - \phi(x_2) = \Delta_2,$$

$$x_2 - x_1 = \delta, \qquad X - x_1 = h.$$

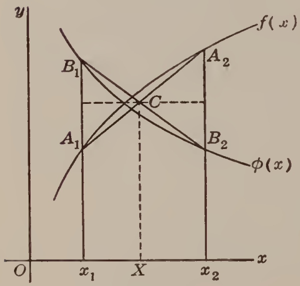

FIG. 55

From the figure, the triangles A_1CB_1 and A_2CB_2 are similar, and

$$A_1B_1 = \Delta_1, \qquad A_2B_2 = \Delta_2.$$

Their altitudes, when C is taken as the vertex, are respectively h and $\delta - h$. Hence

$$\frac{h}{\Delta_1} = \frac{\delta - h}{\Delta_2}.$$

On solving this equation for h we find:

$$h = \frac{\Delta_1}{\Delta_1 + \Delta_2}\,\delta. \tag{3}$$

If $f(x_1) > \phi(x_1)$ and $f(x_2) < \phi(x_2)$, the result still holds, for Δ_1 and Δ_2 now become negative, but their numerical values

correspond to the lengths of the sides of the triangles in question.

It is easy to express in words the result embodied in 3).

RULE. *In order to see what fraction of* $\delta = x_2 - x_1$ *must be added to* x_1 *in order to give* X, *form the differences*

$$\phi(x_1) - f(x_1), \qquad f(x_2) - \phi(x_2).$$

Then the fraction is the quotient of the first of these differences by their sum.

In practice, an accurately drawn figure on a large scale will often afford a quicker and sufficiently accurate solution.

Example. Returning to Ex. 1 above, we have:

$$f(x) = \cos x, \qquad \phi(x) = x;$$

$$\delta = x_2 - x_1 = .0029, \qquad x_1 = .7389, \qquad x_2 = .7418.$$

$$\phi(x_1) - f(x_1) = -.0004; \qquad f(x_2) - \phi(x_2) = -.0045.$$

$$\frac{.0004}{.0049}.0029 = \frac{.0116}{49} = .0002.$$

Hence the value of the new approximation is

$$X = .7389 + .0002 = .7391.$$

The student will have no difficulty in completing Ex. 2 above in a similar manner. It turns out that the correction is here less than one tenth of δ, and hence it does not influence the second place of decimals: $x' = 1.30$.

Newton's Method. If a higher degree of accuracy is desired, it is well now to apply Newton's method to the function

$$F(x) = f(x) - \phi(x).$$

In the case of Ex. 1 above it is pretty clear that we already have four-place accuracy, and the computation of $F(x)$ for the value $X = .7391$ would only verify the result. This is as far as we can go with four-place tables. If we needed greater

accuracy, we should use Newton's method and five or six-place tables.

Example 2 has been carried only to two-place accuracy, or three significant figures. We can obtain two further figures with the tables at our disposal.

$$y = F(x) = \tan x - e^x.$$

$$y_1 = F(1.30) = 3.602 - 3.669 = -.067.$$

$$\frac{dy}{dx} = \sec^2 x - e^x, \qquad \left.\frac{dy}{dx}\right|_{x=1.30} = 13.97 - 3.67 = 10.30$$

$$X = 1.30 + \frac{.067}{10.3} = 1.3067.$$

To test this result, however, would require five-place tables.

EXERCISES

Solve the following equations:

1. $\cot x = x, \qquad 0 < x < \pi.$

2. $e^x + \log x = 1.$

3. The hyperbolic sine (sh x or sinh x) and cosine (ch x or cosh x) are defined as follows:

$$\operatorname{sh} x = \frac{e^x - e^{-x}}{2}, \qquad \operatorname{ch} x = \frac{e^x + e^{-x}}{2},$$

and are tabulated in Peirce's Tables, pp. 120–123. By means of these, reduce the treatment of Ex. 2, § 4, to the methods of the present paragraph.

6. Successive Approximations. We come now to one of the most important of all the methods of numerical computation. In physics it is known as the method of Trial and Error; in mathematics it goes under the name of the method of Successive Approximations.

The problem is that of solving a pair of simultaneous equations,

1) $\qquad F(x, y) = 0, \qquad \Phi(x, y) = 0.$

The cases which arise in practice are characterized in general by two things: First, there is only one solution of the equations which interests us, and the physical problem enables us to make a fairly good guess at it for the first approximation. Secondly, each of the equations 1) is simple, the curve can readily be plotted in character, and the equation can be solved with ease numerically for the dependent variable when a numerical value has been given to the independent variable. But elimination of one of the unknowns, though sometimes possible, is not expedient, since the resulting equation is hard to solve.

The method is as follows. Plot the curves 1) in character with sufficient accuracy to determine which of them is steeper (*i.e.* has the numerically larger slope) at their point of intersection. Let

C_1: $\qquad F(x, y)=0 \qquad$ or $\qquad y=f(x)$

be the one that is less steep,

C_2: $\qquad \Phi(x, y)=0 \qquad$ or $\qquad x=\phi(y)$,

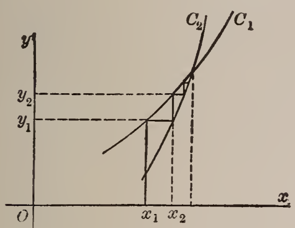

FIG. 56

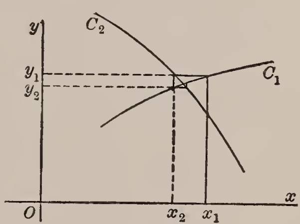

FIG. 57

the other. Then, making the best guess we can to start with, $x=x_1$, compute y_1 from the equation of C_1:

$$y_1=f(x_1),$$

and substitute this value in the equation of C_2, thus getting the second approximation:

$$x_2=\phi(y_1).$$

Proceeding with x_2 in the same manner, we obtain first y_2, then x_3, and so on.

The successive steps of the process are shown geometrically by the broken lines of the figures.

The success of the method depends on the ease with which y can be determined when x is given in the case of C_1, while for C_2 x must be easily attainable from y. If the curves happened to have slopes numerically equal but opposite in sign, the process would converge slowly or not at all. But in this case the arithmetic mean of x_1 and x_2 will obviously give a good approximation.

The method has the advantage that each computation is independent of its predecessor. An error, therefore, while it may delay the computation, will not vitiate the result.

Example. A beam 1 ft. thick is to be inserted in a panel 10×15 ft. as shown in the figure. How long must the beam be made?

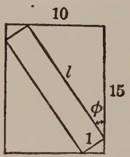

FIG. 58

We have:

$$\begin{cases} \sin\phi + l\cos\phi = 15, \\ \cos\phi + l\sin\phi = 10. \end{cases}$$

Hence $$\cos^2\phi - \sin^2\phi = 10\cos\phi - 15\sin\phi.$$

Now an expression of the form

$$a\cos\phi - b\sin\phi$$

can always be written as

$$\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos\phi - \frac{b}{\sqrt{a^2+b^2}}\sin\phi\right) = \sqrt{a^2+b^2}\cos(\phi+\alpha),$$

where $$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \qquad \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

In the present case, then,

$$\cos 2\phi = \sqrt{325}\cos(\phi+\alpha),$$

where $$\cos\alpha = \frac{10}{\sqrt{325}}, \qquad \sin\alpha = \frac{15}{\sqrt{325}}.$$

Thus α is an angle of the first quadrant and

$$\tan\alpha = \tfrac{3}{2}, \qquad \alpha = 56^\circ\ 16'.$$

Our problem may be formulated, then, as follows: To find the abscissa of the point of intersection of the curves:

$$y = \cos 2\phi, \qquad y = \sqrt{325}\cos(\phi + \alpha).$$

We know from the figure a good approximation to start with, namely:

$$\tan\phi = \tfrac{2}{3}, \qquad \phi = 33^\circ\ 44'.$$

For this value of ϕ the slopes are given by the equations: *

$$\frac{180}{\pi}\cdot\frac{dy}{d\phi} = -2\sin 2\phi = -2\sin 67^\circ\ 28' = -1.8,$$

$$\frac{180}{\pi}\cdot\frac{dy}{d\phi} = -\sqrt{325}\sin(\phi+\alpha) = -\sqrt{325} = -18.$$

Hence we have:

C_1: $$y = \cos 2\phi;$$

C_2: $$y = \sqrt{325}\cos(\phi + \alpha) \quad \text{or} \quad \phi = \cos^{-1}\frac{y}{\sqrt{325}} - \alpha.$$

Beginning with the approximation

$$\phi_1 = 33^\circ\ 44',$$

we compute $$y_1 = \cos 67^\circ\ 28' = .3832.$$

Passing now to the curve C_2, we compute its ϕ when its $y = y_1$:

$$.3832 = \sqrt{325}\cos(\phi_2 + \alpha), \qquad \phi_2 = 32^\circ\ 31'.$$

We now repeat the process, beginning with $\phi_2 = 32^\circ\ 31'$ and find:

$$y_2 = \cos 65^\circ\ 02' = .4221,$$

$$.4221 = \sqrt{325}\cos(\phi_3 + \alpha), \qquad \phi_3 = 32^\circ\ 23'.$$

A further repetition gives $\phi_4 = 32^\circ\ 22'$, and this is the value of the root we set out to determine.

* Since the degree is here taken as the unit of angle, the formulas of differentiation involve the factor $\pi/180$; cf. Chap. V, § 2.

EXERCISES

1. Solve the same problem for a beam 2 ft. thick.

2. A cord 1 ft. long has one end fastened at a point O 2 ft. above a rough table, and the other end is tied to a rod 2 ft. long. How far can the rod be displaced from the vertical through O and still remain in equilibrium when released?

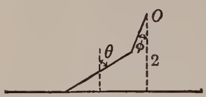

FIG. 59

The equations on which the solution depends are:

$$\begin{cases} 2\cot\theta + \dfrac{1}{\mu} = \cot\phi, \\ 2\cos\theta + \cos\phi = 2. \end{cases}$$

If the coefficient of friction $\mu = \frac{1}{2}$, find the value of ϕ.

3. A heavy ring can slide on a smooth vertical rod. To the ring is fastened a weightless cord of length $2a$, carrying an equal ring knotted at its middle point and having its further end made fast at a distance a from the rod. Find the position of equilibrium of the system.

4. Solve Example 2, § 4, by the method of successive approximations.

7. Arrangement of the Numerical Work in Tabular Form. In the foregoing paragraphs we have laid the chief stress on setting forth the great ideas which underlie these powerful methods of numerical computation. There are, however, certain details of technique which are important, not only for ease in keeping in view the results obtained, but also for accuracy, since they reduce the numerical work to a system. We will illustrate what we mean by an example.

Example. Let it be required to find all the values of x between 0° and 360° which satisfy the equation

$$\sin x = \log_{10}(1 - \cos x).$$

A free-hand graph of each of the functions

1) $$y = \sin x, \qquad y = \log_{10}(1 - \cos x)$$

shows that there is one root between 0° and 180° and a second between 180° and 360°. But these roots cannot be located with any great accuracy in this manner. It is necessary to do exact table work, and to keep the successive results in such form that they are convenient for later reference.

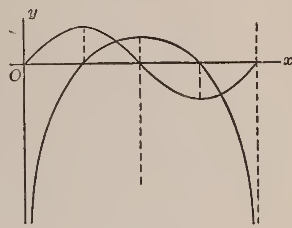

FIG. 60

To this end such a table as the following is useful.* Begin with the trial value $x = 150°$.

x	150°				
$\cos x$	$-$ .8660				
$1 - \cos x$	1.8660				
$\log_{10}(1 - \cos x)$	.2709				
$\sin x$	.5000				

Since the ordinate of the sine curve is larger than that of the logarithmic curve, it is clear from the figure that x is too small. Try $x = 160°$.

Before proceeding further let us ask ourselves whether the above scheme is the simplest for the example in hand. For the special value $x = 150°$ we know $\cos x$ without reference to the tables, and hence one entry of the tables was sufficient. But when $x = 160°$, it will be necessary to enter the tables first for $\cos x$, a second time for $\log_{10}(1 - \cos x)$, and still a third time for $\sin x$.

* Paper ruled in small squares is convenient for these tables, the individual digits being written in separate squares.

Now, $$1 - \cos x = 2 \sin^2 \frac{x}{2},$$

$$\log_{10}(1 - \cos x) = \log_{10} \sin^2 \frac{x}{2} + \log_{10} 2$$

$$= 2 \log \sin \frac{x}{2} + .3010.$$

Hence it is possible to get along with only two entries of the tables if we make use of the following scheme.

x	$160°$	$164°$	$163° 3'$
$\frac{1}{2}x$	$80°$	$82°$	$81° 32'$
$\log_{10} \sin \frac{1}{2} x$	$\bar{1}.9934$	$\bar{1}.9958$	$\bar{1}.9952$
$2 \log_{10} \sin \frac{1}{2} x$	$\bar{1}.9868$	$\bar{1}.9916$	$\bar{1}.9904$
$+ .3010$	.2878	.2926	.2914
$\sin(180 - x)$	.3420	.2756	.2916

The ordinate of the sine curve is still in excess, but only slightly so. Try $x = 164°$. It is seen that the curves have now crossed. Moreover, the two approximations for x — namely, $160°$ and $164°$ — are so near together that we can with advantage apply the method of interpolation of § 5. We have:

$$\phi(x) = \sin x, \qquad f(x) = \log_{10}(1 - \cos x);$$

$$x_1 = 160°, \qquad x_2 = 164°, \qquad \delta = 4°;$$

$$\phi(x_1) = .3420, \qquad f(x_1) = .2878, \qquad \Delta_1 = .0542;$$

$$\phi(x_2) = .2756, \qquad f(x_2) = .2926, \qquad \Delta_2 = .0170;$$

$$h = \frac{\Delta_1}{\Delta_1 + \Delta_2} \delta = \frac{.0542}{.0712} 4 = 3.05.$$

Thus the correction is seen to be $3.05°$, or $3° 3'$, and the new approximation is:

$$x = 163° 3'.$$

For this value of x the values of the two functions, $f(x)$ and $\phi(x)$, differ by a quantity which is comparable with the error of the tables, and the problem is solved.

EXERCISES

1. Determine the other root in the above problem.

2. Solve the equation:

$$\cot x = \log_{10}(1 + \sin x), \qquad 0 < x < 90°.$$

3. Find the positive root of the equation

$$e^{-x} = x^3 - x.$$

Suggestion. Tabulate x, x^3 (from a table of cubes), $x^3 - x$, and e^{-x}.

8. Algebraic Equations. By an *algebraic equation* is meant an equation of the form

$$1) \qquad a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \qquad a_0 \neq 0,$$

where n denotes a positive integer.

If the coefficients a_0, a_1, $\cdots$ are numerical, the roots can be approximated to by the method of interpolation or by Newton's method. In either case it becomes necessary to compute the value of the polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

for several values of x, the later ones of which will be at least three- or four-place numbers. There are labor-saving devices for performing these computations, to which we now turn.

Numerical Computation of Polynomials. Let a cubic polynomial, for example, be given:

$$f(x) = ax^3 + bx^2 + cx + d,$$

and let it be required to compute $f(x)$ for the value $x = m$. Write down the following scheme:

$$\begin{array}{cccc} a & am + b & am^2 + bm + c & f(m) \\ \hline am & am^2 + bm & am^3 + bm^2 + cm & \end{array},$$

the explanation of which is as follows. Begin with the first coefficient, a, and multiply it by m to get the expression am which stands below the line. To this expression add the second coefficient, b, to get the second expression above the line, $am + b$. Next, multiply this expression by m to get the expression which stands below it, and continue the process. The last entry above the line will be the required value,

$$f(m) = am^3 + bm^2 + cm + d.$$

Example. Let

$$f(x) = 7x^3 - 6x^2 + 3x - 7,$$

and let it be required to compute the value of $f(x)$ for $x = .8$. Here, the scheme is as follows:

$$\begin{array}{cccc} 7 & -.4 & 2.68 & -4.856 \\ \hline 5.6 & -.32 & 2.144 & \end{array}$$

and hence

$$f(.8) = -4.856.$$

It will be observed that the process requires only additions (or subtractions) and multiplications. The former can be performed mentally. The latter are executed most simply by one of the machines now in general use with computers. These instruments, combined with the method of this paragraph, have rendered Horner's method for solving numerical algebraic equations obsolete.

EXERCISE

Compute the value of

$$5.1x^4 - 3.42x^2 + 1.432x + .8543$$

for $x = .1876$.

In the problems which arise in physics, however, it is not a question of computing all the roots of a numerical equation, about which nothing is known beyond the coefficients. Usually, the equation is a cubic or biquadratic, and only one root is required. Moreover, from the nature of the problem, a close

guess at the value of this root can be made at the outset. Then the methods set forth in this paragraph and in §§ 2, 3 lead quickly to the desired result.

EXERCISES

Solve the following equations, being given that there is one root, and only one, between 0° and 90°:

1. $$4\cos^3\theta - 3\cos\theta = .5283, \qquad 0° < \theta < 90°.$$

2. $$\sin^3\theta - .75\sin\theta = .1278, \qquad 0° < \theta < 90°.$$

3. Find the root of the equation

$$x^4 + 2.6x^3 - 5.2x^2 - 10.4x + 5.0 = 0$$

which lies between 0 and 1.

4. Find the root of the equation

$$3x^4 - 12x^3 + 12x^2 - 4 = 0$$

which lies between 2 and 3.

9. Continuation. Cubics and Biquadratics. Aside from the special problem of numerical computation, the simpler algebraic equations present an intrinsic interest which should not be ignored.

Transformations. a) Let the cubic equation

1) $$f(x) = ax^3 + bx^2 + cx + d = 0, \qquad a \neq 0,$$

be given, and let x be replaced by y, where

2) $$y = x - h, \qquad x = y + h.$$

Then

$$\begin{aligned} f(x) &= a(y+h)^3 + b(y+h)^2 + c(y+h) + d = \phi(y) \\ &= ay^3 + (3ah + b)y^2 + \cdots, \end{aligned}$$

where the later coefficients are easily written down.

If $y = \beta$ is a root of the equation

3) $$\phi(y) = 0,$$

then $$x = \beta + h$$

will be a root of equation 1). For, it is always true that

$$f(x) = \phi(y)$$

when x and y are connected by the relation 2).

Here, h is any number we please. In particular, h can always be so chosen that the coefficient of the second term of 3) will drop out. It is sufficient to set

4) $$3ah + b = 0, \qquad \text{or} \qquad h = -\frac{b}{3a}.$$

Obviously, the same method can be used to transform an algebraic equation of any degree into a new equation whose second term is lacking.

EXERCISES

Transform the following equations into equations in which the second term is lacking.

1. $x^3 + x^2 - x + 1 = 0.$
2. $3x^3 - 4x^2 + 2 = 0.$
3. $x^4 + x^3 - x^2 + 1 = 0.$
4. $5x^4 - 4x^3 + x^2 + x - 80 = 0.$
5. $3x^4 - 7x^3 + x^2 - x - 1 = 0.$
6. $x^6 + x^5 + x^2 + x + 1 = 0.$

b) Let the equation

5) $$f(x) = x^4 + px^2 + qx + r = 0$$

be given, and let x be replaced by y, where

6) $$y = \frac{x}{k}, \qquad x = ky.$$

Then

$$f(x) = k^4y^4 + k^2py^2 + kqy + r.$$

Denoting this last polynomial by $\phi(y)$, we have

$$f(x) = \phi(y)$$

for all values of x and y which are connected by the relation 6).

It is clear that, if $y=\beta$ is a root of the equation

7) $$\phi(y)=0,$$

then $x=k\beta$ will be a root of 5).

The factor k is arbitrary, and we can always determine it so that, on dividing equation 7) through by k:

$$y^4+\frac{p}{k^2}y^2+\frac{q}{k^3}y+\frac{r}{k^4}=0,$$

the coefficient of y^2 will be numerically equal to unity (provided that $p \neq 0$):

i) $\dfrac{p}{k^2}=1$ or $k=\sqrt{p}$, if $p>0$;

ii) $\dfrac{p}{k^2}=-1$ or $k=\sqrt{-p}$, if $p<0$.

In this way, equation 5) can be reduced to one of the two forms

α) $$y^4+y^2+Ay+B=0;$$

β) $$y^4-y^2+Ay+B=0.$$

If, in particular, $p=0$ and $q \neq 0$, 5) can be reduced to the form

γ) $$y^4+y+B=0.$$

The method can be applied to any algebraic equation whose second term is lacking:

$$x^n+c_2x^{n-2}+c_3x^{n-3}+\cdots+c_n=0.$$

EXERCISES

1. Replace the equation

$$7x^4-175x^2+16x+10=0$$

by an equation of the type β), and state precisely the relation of the roots of the second equation to those of the first.

2. Show that, if in the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0,$$

where $a_0 \neq 0$ and $a_n \neq 0$, the transformation

$$y = \frac{1}{x}$$

is made, the roots of the new equation,

$$a_ny^n + a_{n-1}y^{n-1} + \cdots + a_0 = 0$$

are the reciprocals of the roots of the given equation.

3. If on transforming equation 1) by 2), where h is determined by 4), the constant term in the resulting equation 3), $\phi(y) = 0$, does not vanish, the further transformation

$$8) \qquad y = \frac{1}{z}, \qquad \text{or} \qquad x = \frac{1}{z} + h,$$

will carry 1) into an equation in which the linear term is lacking:

$$Az^3 + Bz^2 + D = 0. \qquad A \neq 0, \quad D \neq 0.$$

The theorem holds in full generality for an algebraic equation of any higher degree. State it accurately.

4. Replace the equation

$$x^4 - 4x^3 - 6x^2 + 16x - 4 = 0$$

by an equation of the type

$$Ay^4 + By^3 + Cy^2 + D = 0.$$

Graphical Treatment. We have already seen that the cubic

$$x^3 + px + q = 0$$

can be solved graphically by cutting the standard graph

$$y = x^3$$

by the straight line,

$$y = -px - q.$$

Since the general cubic can be reduced by the transformation 2) to a cubic of this type, we may consider the general problem of the graphical solution of a cubic as solved.

To obtain a similar solution for the general biquadratic,

9) $$ax^4 + bx^3 + cx^2 + dx + e = 0, \qquad a \neq 0,$$

begin by reducing it to one of the three forms:

i) $$y^4 + y^2 + Ay + B = 0;$$

ii) $$y^4 - y^2 + Ay + B = 0;$$

iii) $$y^4 \qquad + Ay + B = 0.$$

An equation of type i):

$$x^4 + x^2 + Ax + B = 0,$$

can be solved graphically by cutting the standard curve

$$y = x^4$$

by the parabola

$$y = -x^2 - Ax - B.$$

A similar procedure leads to a solution in the case of each of the other two types, ii) and iii).

The Method of Curve Plotting. Let the coefficients a, e in equation 9) be different from 0. By means of Ex. 3, p. 192, the equation can be reduced to one of the following type:

$$Ax^4 + Bx^3 + Cx^2 + E = 0.$$

In order to discuss the number and location of the roots of this equation, it is sufficient to plot the curve

$$y = Ax^4 + Bx^3 + Cx^2 + E.$$

Since all the maxima, minima, and points of inflection of this curve can be determined by means, at most, of quadratic equations, the problem is readily solved in any given numerical case.

EXERCISES

Determine the number of real roots of each of the following equations, and locate them approximately.

1. $3x^4 + 8x^3 - 90x^2 + 100 = 0.$

2. $3x^4 + 8x^3 - 90x^2 + 500 = 0.$

3. $3x^4 + 8x^3 - 90x^2 + 1500 = 0.$

4. Show that the equation

$$3x^4 + 4x^3 + 2x^2 + 1 = 0$$

has no real roots.

How many real roots has each of the following equations?

5. $x^5 - 5x - 1 = 0.$

6. $x^3 + 7x - 1 = 0.$

7. $x^3 - 4x + 1 = 0.$

8. $x^3 - 3x - 2 = 0.$

9. $x^3 - x + 3 = 0.$

10. $4x^3 - 15x^2 + 12x + 1 = 0.$

11. $3x^4 + 4x^3 + 6x^2 - 1 = 0.$

12. $3x^4 - 4x^3 + 12x^2 + 7 = 0.$

13. How many positive roots has the equation

$$6x^4 + 8x^3 - 12x^2 - 24x - 1 = 0?$$

14. Has the equation

$$3x^8 - 8x^6 + 12x^3 + 1 = 0$$

any real roots?

15. By means of the graph of the function

$$y = x^3 + px + q$$

show that the equation

$$x^3 + px + q = 0$$

has

(*a*) 1 real root when $\dfrac{p^3}{27} + \dfrac{q^2}{4} > 0;$

(*b*) 3 real roots when $\dfrac{p^3}{27} + \dfrac{q^2}{4} < 0;$

(*c*) 2 real roots when $\dfrac{p^3}{27} + \dfrac{q^2}{4} = 0,$ $\{p \text{ and } q \text{ not both } 0\}$

(*d*) 1 real root when $\dfrac{p^3}{27} + \dfrac{q^2}{4} = 0,$ $\{p = q = 0\}$

In case (*c*) it is customary to count one of the roots twice; in case (*d*), to count the root three times.

16. Extend the criterion of Ex. 15 to the case of the general cubic

$$ax^3 + bx^2 + cx + d = 0.$$

10. Curve Plotting. We will close this chapter by considering the application of the principles set forth in the earlier paragraph on curve plotting (Chap. III, § 5) to some interesting curves of a more complex nature.

Example 1. To plot the curve

$$1) \qquad y = \frac{1}{x-1} + \frac{1}{x+1}.$$

The curve is obviously not symmetric in either axis; but the test for symmetry in the origin is fulfilled, since on replacing x by $-x$ and y by $-y$ the new equation,

$$-y = \frac{1}{-x-1} + \frac{1}{-x+1}$$

is equivalent to the original equation, 1). Incidentally we observe that the curve passes through the origin.

In consequence of the symmetry just noted it will be sufficient to plot the curve for positive values of x and then rotate the figure about the origin through 180°.

To each positive value of x but one there corresponds one value of y. When x approaches 1 as its limit from above (*i.e.* always remaining greater than 1), y becomes positively infinite. Hence the line $x = 1$ is an asymptote for one branch of the curve.

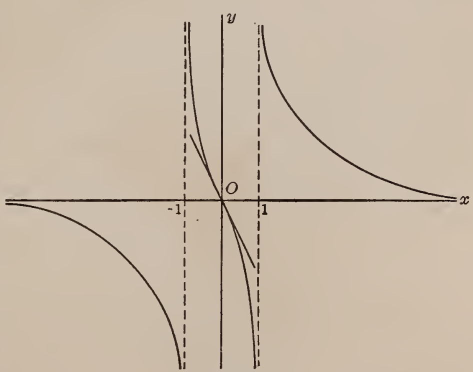

FIG. 61

When x approaches 1 from below, y becomes negatively infinite, and hence this same line, $x=1$, is an asymptote for a second branch of the curve.

For all other positive values of x, y is continuous.

The slope of the curve is given by the equation

$$2) \qquad \frac{dy}{dx} = -\left(\frac{1}{(x-1)^2} + \frac{1}{(x+1)^2}\right),$$

and is seen to be negative for all values of x for which y is continuous. Thus, in particular, the curve is seen to have no maxima or minima, or in fact any points at which the tangent is horizontal.

The second derivative is given by the formula

$$3) \qquad \frac{d^2y}{dx^2} = 2\left(\frac{1}{(x-1)^3} + \frac{1}{(x+1)^3}\right).$$

When $x > 1$, the right-hand side of this equation is always positive, and so the curve is concave upward in this interval. Moreover, it is evident from 1) that, when $x = +\infty$, y approaches 0 from above, and so the positive axis of x is also an asymptote.

In the interval $0 < x < 1$, the second derivative is surely sometimes negative, for this is obviously the case when x is only slightly less than 1. Is d^2y/dx^2 always negative in this interval? If not, it must pass through the value 0; for a continuous function cannot change from a positive to a negative value without taking on the intermediate value 0.* Let us set, then, the right-hand side of equation 3) equal to 0 and solve:

$$2\left(\frac{1}{(x-1)^3} + \frac{1}{(x+1)^3}\right) = 0.$$

* How must the graph of a continuous function look, which is sometimes positive and sometimes negative? It must cross the axis of abscissas, must it not? At the point or points where it crosses, the function has the value 0.

This equation is equivalent to the following:

$$\frac{1}{(x-1)^3} = -\frac{1}{(x+1)^3}.$$

Extracting the cube root of each side of this equation, we have:

$$\frac{1}{x-1} = -\frac{1}{x+1}.$$

Clearing of fractions we find:

$$x+1 = -(x-1),$$

or

$$2x = 0.$$

Hence $x=0$ is the only value of x for which d^2y/dx^2 can vanish, and we see at once that the right-hand side of 3) does vanish for $x=0$.

We have thus proven that the continuous function 3) is nowhere 0 in the interval $0 < x < 1$, and since it is negative in part of this interval, it is negative throughout. Hence the curve is concave downward throughout the interval.

It is now easy to complete the graph. The curve has one point of inflection, — namely, the origin, — and the slope there is, by 2), equal to -2.

EXERCISES

Plot the following curves:

1. $y = \dfrac{3}{3+x^2}$.
2. $y = \dfrac{3x}{3+x^2}$.
3. $y = \dfrac{1}{x-2} + \dfrac{1}{x+2}$.
4. $y = \dfrac{1}{x} + \dfrac{1}{x-1}$.
5. $y = \dfrac{1}{x} + \dfrac{1}{x+1}$.
6. $y = \dfrac{1}{x^2-1}$.
7. $y = \dfrac{1}{x^2}$.
8. $y = \dfrac{1}{(1-x)^2}$.
9. $y = \dfrac{1}{x^3}$.
10. $y = \dfrac{1}{(x+1)^3}$.

11. $y = x + \dfrac{1}{x}$.

12. $y = x - \dfrac{1}{x}$.

13. $y = \dfrac{4}{1-x} + x^2 - 2x$.

14. $y = \dfrac{3}{3+x} - 6x - x^2$.

15. $y = \dfrac{1}{x-1} - \dfrac{1}{x+1}$.

16. $y = \dfrac{1}{x} - \dfrac{1}{x-1}$.

Example 2. To plot the curve

4) $$y^2 = x^2 + x^3.$$

We observe first of all that the curve is symmetric in the axis of x. It is sufficient, therefore, to plot the curve for positive values of y, and then fold this part of the curve over on the axis of x. The curve goes through the origin.

Unlike the examples hitherto considered, this curve does not permit an arbitrary choice of x. It is only when the right-hand side is positive or zero, *i.e.* when

$$x^2 + x^3 \geqq 0,$$

or

$$x^2(1+x) \geqq 0 \qquad \text{or} \qquad x \geqq -1,$$

that there will be a corresponding value of y and thus a point with the given abscissa.

Obviously, the curve cuts the axis of x at the origin and at the point $x = -1$. We have, then, essentially two problems:

i) to plot the curve for $x > 0$;
ii) to plot the curve for $-1 < x < 0$.

i) When $x > 0$, the positive value of y is given by the equation

5) $$y = x\sqrt{1+x}.$$

Hence

6) $$\frac{dy}{dx} = \frac{2+3x}{2\sqrt{1+x}}.$$

For positive values of x the right-hand side of this equation is always positive, and hence there are no horizontal tangents in the interval under consideration; the slope of this part of the curve is always positive. In particular, the slope at the origin is unity:

$$\left.\frac{dy}{dx}\right|_{x=0} = 1.$$

The second derivative has the value

7) $$\frac{d^2y}{dx^2} = \frac{4+3x}{4(1+x)^{\frac{3}{2}}}.$$

The right-hand side of this equation is always positive in this interval, and thus it appears that the curve is concave upward for all positive values of x.

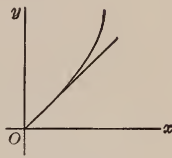

FIG. 62

ii) When $-1 < x < 0$, the positive value of y is no longer given by the formula 5), since x is now negative.* In the present case,

8) $$y = -x\sqrt{1+x},$$

and consequently

9) $$\frac{dy}{dx} = -\frac{2+3x}{2\sqrt{1+x}},$$

10) $$\frac{d^2y}{dx^2} = -\frac{4+3x}{4(1+x)^{\frac{3}{2}}}.$$

The first derivative will vanish if, and only if,

$$2 + 3x = 0,$$

or $$x = -\tfrac{2}{3}.$$

* The student must have clearly in mind the definition of the function expressed by the $\sqrt{\ }$ sign, which was laid down in Chap. I, § 1. This function is the *positive* square root of the radicand; it can never take on a negative value.

It is, therefore, important to determine the corresponding point on the curve and draw the tangent there:

$$y\Big|_{x=-\frac{2}{3}} = -(-\tfrac{2}{3})\sqrt{1-\tfrac{2}{3}} = \frac{2\sqrt{3}}{9} = .38.$$

Two other important points for the present curve are the origin and the point $x=-1$, $y=0$. At these points the slope has the following values:

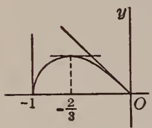

FIG. 63

$$\left.\frac{dy}{dx}\right|_{x=0} = -1; \qquad \left.\frac{dy}{dx}\right|_{x=-1} = \infty.$$

Draw the corresponding tangents.

From the expression 10) for the second derivative it is clear that, when $-1 < x < 0$, the right-hand side of this equation is always negative, and hence the curve is concave downward throughout the whole interval in question. We can now draw in the curve in this interval, Fig. 63.

The curve is now complete above the axis of x. It remains, therefore, merely to fold this part over on that axis. The entire curve is shown in Fig. 64.

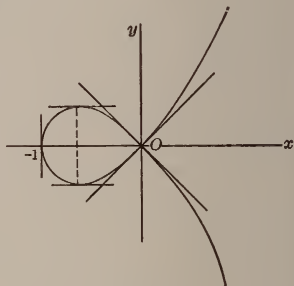

FIG. 64

EXERCISES

Plot the following curves:

1. $y^2 = x^2 - x^3$.

2. $y^2 = x - 2x^2 + x^3$.

3. $$y^2 = (x-a)^2(Ax+B)$$

Suggestion: Write the second factor in the form

$$Ax + B = A(x - b), \text{ where } b = \frac{B}{A},$$

and make two cases: i) $A > 0$; ii) $A < 0$. Discuss the omitted case, $A = 0$.

4. $y^2 = x^2 - x^4$. 5. $y^2 = x^2 + x^4$.

Example 3. To plot the curve

11) $$y^2 = x(x-1)(x-2).$$

The curve lies wholly in the regions

$$0 \leqq x \leqq 1 \quad \text{and} \quad 2 \leqq x.$$

It is symmetric in the axis of x, and hence it is sufficient to plot it for positive values of y.

The function

$$y = \sqrt{x(x-1)(x-2)}$$

is continuous in the interval $0 \leqq x \leqq 1$. It starts with the value 0 when $x = 0$, increases, and finally decreases to 0 when $x = 1$.

When x, starting with the value 2, increases, y, starting with the value 0, increases, always remaining positive, and increasing without limit as x becomes infinite.

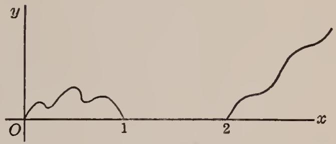

Fig. 65

So much from considerations of continuity. A more specific discussion of the character of the curve can be given by means of the derivatives of the function.

The slope is given by the formula

12) $$2y\frac{dy}{dx} = 3x^2 - 6x + 2$$

or

13) $$\frac{dy}{dx} = \frac{3x^2 - 6x + 2}{2\sqrt{x(x-1)(x-2)}}.$$

The slope is infinite when $x = 0$ or 1:

$$\left.\frac{dy}{dx}\right|_{x=0} = \infty, \qquad \left.\frac{dy}{dx}\right|_{x=1} = \infty.$$

At these points, the tangent is vertical.

The slope is 0 when

$$3x^2 - 6x + 2 = 0.$$

The roots of this equation are

$$x = 1 + \frac{1}{\sqrt{3}}, \qquad x = 1 - \frac{1}{\sqrt{3}}.$$

The first of these values does not correspond to any point on the curve. The second, $x = .42$, yields a horizontal tangent, the ordinate being

$$y = \sqrt{\frac{2}{3\sqrt{3}}} = .62.$$

Plot this point and draw the tangent. From the above discussion on the basis of continuity it is obvious that this point must be a maximum, and we see that there are no other maxima or minima. But it is not clear that the curve has no points of inflection in this interval.

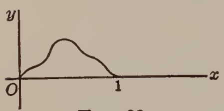

Fig. 66

To treat this question, compute the second derivative. This might be done by means of formula 13); but it is simpler to use 12):

$$2y\frac{d^2y}{dx^2} + 2\frac{dy^2}{dx^2} = 6x - 6,$$

$$y\frac{d^2y}{dx^2} = 3x - 3 - \frac{dy^2}{dx^2}.$$

Substitute here the value of dy/dx from 13) and reduce:

14) $$y\frac{d^2y}{dx^2} = \frac{3x^4 - 12x^3 + 12x^2 - 4}{4x(x-1)(x-2)}.$$

And now we seem to be in difficulty. How are we going to tell when d^2y/dx^2 is positive, when negative?

First of all, y is positive, and so the sign of d^2y/dx^2 will be the same as that of the right-hand side of the equation.

Secondly, in the interval in question, $0 < x < 1$, the denominator is positive.

All turns, then, on whether the numerator, *i.e.* the function

15) $$u = 3x^4 - 12x^3 + 12x^2 - 4,$$

is positive or negative. To answer this question, plot the graph of the function 15). The slope of the graph is given by the equation

16) $$\frac{du}{dx} = 12x^3 - 36x^2 + 24x = 12x(x-1)(x-2).$$

In the interval in question, the right-hand side of this last equation is always positive. Hence u increases with x throughout the interval $0 \leqq x \leqq 1$, and consequently attains its greatest value at the end-point, $x = 1$. Here,

$$u\Big|_{x=1} = -1.$$

We see, therefore, that u is negative throughout the whole interval in question, and consequently the graph of 1) is concave downward in this interval.

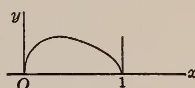

FIG. 67

The reasoning by which we determined whether u is positive or negative is an excellent illustration of the practical application of the methods of curve plotting which we have learned. It is in no wise a question of the precise values of u which correspond to x. The question is merely: Is u positive, or is it negative? Without the labor of a single computation involving table work we have answered this question with the greatest ease. Such questions as these arise again and again in physics, and the aid which the calculus is able to render here is most important.

One further point. It may seem to have been a fluke that we were able to factor the polynomial in 16) and thus simplify so materially the further discussion. And yet, in the problems which arise in practice, — the problems with a *pedigree*, — just such simplifications as this present themselves with great frequency.

To complete the graph, it remains to consider the interval $2 \leqq x < \infty$. Since

$$\left.\frac{dy}{dx}\right|_{x=2} = \infty,$$

the tangent to the curve is vertical at the point where the curve meets the axis of x. It is clear, then, that the curve must be concave downward for a while, and so $d^2y/dx^2 < 0$ for values of x slightly greater than 2. This is verified from 14), since

17) $$u|_{x=2} = -4.$$

On the other hand, when x is large, u is positive and d^2y/dx^2 is positive. Hence the curve is concave upward. There must be, therefore, a point of inflection in the interval, and there may be several.

From 14) we see that the second derivative will vanish when and only when

$$3x^4 - 12x^3 + 12x^2 - 4 = 0.$$

The problem is, then, to determine the number of roots of this equation which are greater than 2, and to compute them. Again, it is a question of the graph of 15). When $x > 2$, we see from 16) that

$$\left.\frac{du}{dx}\right|_{x>2} > 0.$$

Hence u steadily increases with x. Now, from 17), u starts with a negative value, and u is positive and large when x is large. Hence u vanishes for just one value of x which is greater than 2. Since $u|_{x=3} = 23$, this root is seen to lie between 2 and 3. It can be determined to any required degree of accuracy by the foregoing methods of

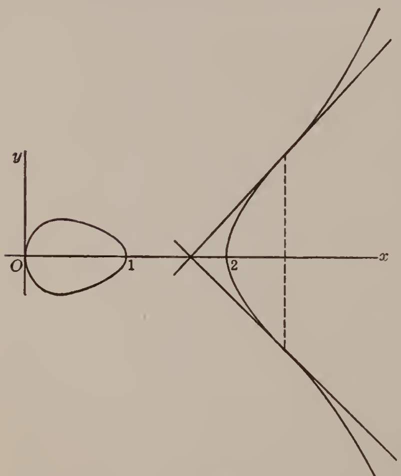

FIG. 68

this chapter, which find herewith a practical application. To two places of decimals it is 2.47.

EXERCISES

Plot the following curves:

1. $y = x^3 - x$.
2. $y = x - x^3$.
3. $y^2 = x^3 + x$.
4. $y^2 = 1 - x^4$.
5. $y^2 = (x^2 - 1)(x^2 - 4)$.
6. $y^2 = (1 - x^2)(x^2 - 4)$.
7. $y^2 = \dfrac{1}{x^3 - x}$.
8. $y^2 = \dfrac{1}{(x^2 - 1)(x^2 - 4)}$.
9. $y^2 = \dfrac{x}{1 - x}$.
10. $y^2 = \dfrac{x}{1 + x}$.
11. $y^2 = \dfrac{x^2}{1 + x^2}$.
12. $y^2 = \dfrac{x^2}{1 - x^2}$.
13. $y^2 = \dfrac{x^2}{x - 1}$.
14. $y^2 = \dfrac{x^2}{1 + x}$.
15. $y^2 = x^3 - 4x^2 + 3x$.
16. $y = \sin x + \sin 2x$.
17. $y = \sin x - \sin 2x$.
18. $y = \cos x + \cos 2x$.
19. $y = \cos x - \cos 2x$.
20. $y = x + \sin x, 0 \leqq x \leqq \pi$.

CHAPTER VIII

THE INVERSE TRIGONOMETRIC FUNCTIONS

1. Inverse Functions. Let

$$(1) \qquad y = f(x)$$

be a given function of x, and let us solve this equation for x as a function of y:

$$(2) \qquad x = \phi(y).$$

Then $\phi(y)$ is called the *inverse function*, or the *inverse* of the function $f(x)$. Thus if $f(x) = x^3$, we have

$$y = x^3.$$

Hence $$x = \sqrt[3]{y},$$

and $\phi(y)$ is here the function $\sqrt[3]{y}$.

When the given function is tabulated, the table also serves as a tabulation of the inverse function. It is necessary merely to enter it from the opposite direction. Thus, if we have a table of cubes, we can use it to find cube roots by simply reversing the rôles of the two columns.

In the same way, the graph of the function (1) serves as the graph of the function (2), provided in the latter case we take y as the *independent variable*, and x as the *dependent variable*, or *function*.

The graph of the inverse function, plotted with x as the independent variable, can be obtained from the former graph as follows. Make the transformation of the plane which is defined by the equations:

$$(3) \qquad \left.\begin{aligned} x' &= y, \\ y' &= x, \end{aligned}\right\} \qquad \text{or} \qquad \left.\begin{aligned} x &= y', \\ y &= x'. \end{aligned}\right\}$$

It is easy to interpret this transformation. Any point, whose coordinates are (x, y), is carried over into a point (x', y') situated as follows: Draw a line L through the origin bisecting the angle between the positive axes of coordinates. Drop a perpendicular from (x, y) on L and produce it to an equal distance on the other side of L. The point thus determined is the point (x', y'). The proof of this statement is immediately evident from the figure.

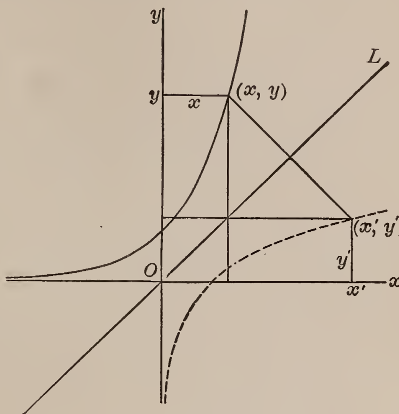

FIG. 69

Thus it appears that the transformation (3) can be generated by rotating the plane about L through 180°.

The transformation is also spoken of as a *reflection in L*, since if a plane mirror were set at right angles to the plane of (x, y) and so that the line L would lie in the surface of the mirror, the image of any figure, as seen in the mirror, would be the transformed figure.

Monotonic Functions. A function, $f(x)$, is said to be *monotonic* if it is single-valued and if, as x increases, $f(x)$ always increases, or else always decreases. We shall be concerned only with functions which are, in general, continuous. It is obvious that the inverse of a monotonic function is also monotonic.

A given function,

$$y = f(x),$$

can in general be considered as made up of a number of pieces, each of which is monotonic in a certain interval.* Thus the function

(4) $$y = x^2$$

* There are functions which do not have this property; but they do not play an important rôle in the elements of the Calculus.

can be taken as made up of two pieces, corresponding respectively to those portions of the graph which lie in the first and the second quadrants, the corresponding intervals for x being here

$$-\infty < x \leqq 0, \qquad 0 \leqq x < \infty.$$

Each of the pieces, of which $f(x)$ is made up, has a monotonic inverse, and thus the function $\phi(x)$ inverse to $f(x)$ is represented by a number of monotonic functions.

In the example just cited, the inverse function is multiple-valued:

$$(5) \qquad y = \pm\sqrt{x}.$$

But one of the two pieces into which the original function was divided yields the single-valued function

$$(6) \qquad y = \sqrt{x},$$

the so-called *principal value* of the multiple-valued function (5); the other,

$$y = -\sqrt{x},$$

the remainder of (5).

The derivative of a monotonic function cannot change sign; but it can vanish or become infinite at special points. Thus

$$y = \sqrt{a^2 - x^2}, \qquad 0 \leqq x \leqq a,$$

is a decreasing monotonic function. Its derivative is, in general, negative; but when $x = 0$, it vanishes, and when $x = a$, it becomes infinite.

Differentiation of an Inverse Function. The function $\phi(x)$ inverse to a given function $f(x)$ can be differentiated as follows. By definition, the two equations

$$(7) \qquad y = \phi(x) \qquad \text{and} \qquad x = f(y)$$

are equivalent; they are two forms of one and the same relation between the variables x and y. Their graphs are identical.

Take the differential of each side of the second equation:

$$dx = df(y) = D_y f(y) \cdot dy.$$

Hence

$$\frac{dy}{dx} = \frac{1}{D_y f(y)}. \qquad (8)$$

To complete the formula, express the right-hand side of (8) in terms of x by means of (7).

2. The Inverse Trigonometric Functions. The inverse trigonometric functions are chiefly important because of their application in the Integral Calculus. They are defined as follows.

(*a*) *The Function* $\sin^{-1}x$. The inverse of the function

$$y = \sin x \qquad (1)$$

is obtained as explained in § 1 by solving this equation for x as a function of y, and is written:

$$x = \sin^{-1}y, \qquad (1')$$

read "the anti-sine of y."* In order to obtain the graph of the function

$$y = \sin^{-1}x \qquad (2)$$

we have, then, merely to reflect the graph of (1) in the bisector of the angle made by the positive coordinate axes. We are thus led to a multiple-valued function, since the line $x = x'(-1 \leqq x' \leqq 1)$ cuts the graph in more than one point, — in fact, in an infinite number of points. For most purposes of the Calculus, however, it is allowable and advisable to pick

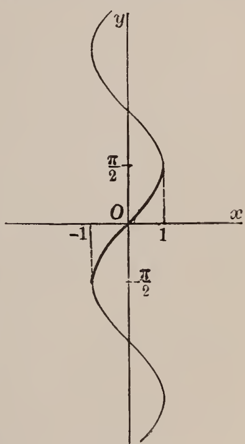

FIG. 70

* The usual notation on the Continent for $\sin^{-1}x$, $\tan^{-1}x$, etc., is arc sin x, arc tan x, etc. It is clumsy, and is followed for a purely academic reason; namely, that $\sin^{-1}x$ might be misunderstood as meaning the minus first power of $\sin x$. It is seldom that one has occasion to write the reciprocal of $\sin x$ in terms of a negative exponent. When one wishes to do so, all ambiguity can be avoided by writing $(\sin x)^{-1}$.

out just *one* value of the function (2), most simply the value that lies between $y = -\pi/2$ and $y = +\pi/2$, and to understand by $\sin^{-1} x$ the *single-valued* function thus obtained. This determination is called the *principal value* of the multiple-valued function $\sin^{-1} x$. Its graph is the portion of the curve in Fig. 70 that is marked by a heavy line. This shall be our convention, then, in the future unless the contrary is explicitly stated, and thus

$$(3) \qquad y = \sin^{-1} x$$

is equivalent to the relations:

$$(3') \qquad x = \sin y, \qquad -\frac{\pi}{2} \leqq y \leqq \frac{\pi}{2}.$$

In particular,

$$\sin^{-1} 0 = 0, \qquad \sin^{-1} 1 = \frac{\pi}{2}, \qquad \sin^{-1}(-1) = -\frac{\pi}{2}.$$

The student should now prepare a second plate, showing the graphs of the three functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$. Place the first in the upper left-hand corner of the sheet; the second, in the upper right-hand corner; and the third on the lower half-sheet. All of these curves can be ruled from the templets. Use a fine lead-pencil; then mark in the principal value of the function in a clean, firm red line. Also mark, in each figure, *all* the principal points, as is done in Fig. 70 of the text.

Differentiation of $\sin^{-1} x$. In order to differentiate the function

$$y = \sin^{-1} x,$$

make the equivalent equation,

$$x = \sin y$$

the point of departure. Then

$$dx = d \sin y = \cos y \, dy.$$

Hence

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

The right-hand side of this equation can be expressed in terms of x as follows. Since

$$\sin^2 y + \cos^2 y = 1$$

and since $\sin y = x$, we have,

$$\cos^2 y = 1 - x^2, \qquad \cos y = \pm\sqrt{1 - x^2}.$$

We have agreed, however, to understand by $\sin^{-1} x$ the principal value of this function. Hence y is subject to the restriction: $-\pi/2 \leqq y \leqq \pi/2$, and consequently $\cos y$ is positive (or zero). We must, therefore, take the upper sign before the radical,* the final result thus being:

(4) $$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}},$$

or

(4′) $$d \sin^{-1} x = \frac{dx}{\sqrt{1 - x^2}}.$$

(*b*) *The Function* $\cos^{-1} x$. The treatment here is precisely similar. The definition is as follows:

(5) $$y = \cos^{-1} x \qquad \text{if} \qquad x = \cos y,$$

(read: "anti-cosine x").

The graph of the function $\cos^{-1} x$ is as shown in Fig. 71. Like $\sin^{-1} x$, this function is also infinitely multiple-valued. A single-valued branch is selected by imposing the further condition

$$0 \leqq y \leqq \pi.$$

This determination is known as the *principal value* of $\cos^{-1} x$:

(6) $$y = \cos^{-1} x, \qquad 0 \leqq y \leqq \pi.$$

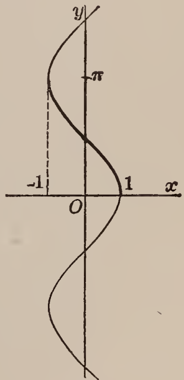

FIG. 71

* Geometrically the slope of the portion of the graph in question is always positive, and so we must use the positive square root of $1 - x^2$.

It will be understood henceforth that the principal value is meant unless the contrary is explicitly stated.

In preparing the graph of this function, mark the principal value as a firm red line.

To differentiate the function $\cos^{-1} x$, use the implicit form of equation (5):

$$x = \cos y.$$

Hence

$$dx = d\cos y = -\sin y \, dy$$

and

$$\frac{dy}{dx} = -\frac{1}{\sin y}.$$

For the principal value, $\sin y$ is positive, and hence

(7) $$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}},$$

or

(7′) $$d\cos^{-1} x = -\frac{dx}{\sqrt{1-x^2}}.$$

The principal values of the functions $\sin^{-1} x$ and $\cos^{-1} x$ are connected by the identical relation:

(8) $$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

By means of this relation, the differentiation of $\cos^{-1} x$ could have been performed immediately.

(c) *The Function* $\tan^{-1} x$. Here, the definition is as follows:

(9) $$y = \tan^{-1} x \qquad \text{if} \qquad x = \tan y,$$

(read: "anti-tangent x").

The principal value is defined as that determination which lies between $-\pi/2$ and $\pi/2$:

(10) $$y = \tan^{-1} x, \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

In preparing the graph of this function, mark the principal value as a firm red line.

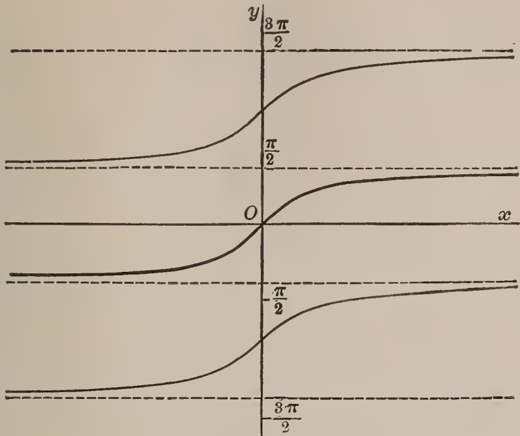

FIG. 72

To differentiate $\tan^{-1} x$ use the implicit form (9). Hence

$$dx = d \tan y = \sec^2 y \, dy,$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}.$$

Since $$\sec^2 y = 1 + \tan^2 y$$

and $\tan y = x$, it follows that

(11) $$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2},$$

or

(12) $$d \tan^{-1} x = \frac{dx}{1 + x^2}.$$

(*d*) *The Function* $\cot^{-1} x$. Here, the definition is:

(13) $$y = \cot^{-1} x \qquad \text{if} \qquad x = \cot y,$$

(read: "anti-cotangent x").

The principal value is chosen as that one which lies between 0 and π:

(14) $$y = \cot^{-1} x, \qquad 0 < y < \pi.$$

The differentiation can be performed as in the case of the function $\tan^{-1} x$, but still more simply by means of the identical relation connecting the principal values of $\tan^{-1} x$ and $\cot^{-1} x$:

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}. \tag{15}$$

Hence

$$\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}, \tag{16}$$

or

$$d \cot^{-1} x = -\frac{dx}{1+x^2}. \tag{17}$$

It is well for the student to make a graph of this function, also, drawing in the principal value, as usual, in red.

The following identity holds for positive values of x, when the principal values of the functions are used:

$$\tan^{-1}\frac{1}{x} = \cot^{-1} x, \qquad 0 < x. \tag{18}$$

For negative values of x it reads:

$$\tan^{-1}\frac{1}{x} = \cot^{-1} x - \pi, \qquad x < 0. \tag{18'}$$

Remarks. The other inverse trigonometric functions, $\sec^{-1} x$, $\csc^{-1} x$, can be treated in a similar manner. They are, however, without importance in practice. Their principal values cannot be defined by means of a single continuous curve. The graph necessarily consists of more than one piece; it is most natural to take it as consisting of two pieces.

Corresponding to the Addition Theorem for each of the trigonometric functions, there are functional relations for the inverse trigonometric functions. Thus, for $\tan^{-1} x$:

$$\tan^{-1} u + \tan^{-1} v = \tan^{-1}\frac{u+v}{1-uv}. \tag{19}$$

These relations, however, are not always true when the principal value of each of the functions is taken, and for this

reason it is usually better not to employ them. If, however, in a particular case, u and v are each numerically less than unity, the principal values can be used throughout in (19).

3. Shop Work. The student will now add to his list of Special Formulas the four new formulas of this chapter. The list of formulas of differentiation is now complete. It reads as follows.

SPECIAL FORMULAS OF DIFFERENTIATION

1. $$dc = 0.$$

2. $$d\,x^n = nx^{n-1}\,dx.$$

3. $$d\sin x = \cos x\,dx.$$

4. $$d\cos x = -\sin x\,dx.$$

5. $$d\tan x = \sec^2 x\,dx.$$

6. $$d\cot x = -\csc^2 x\,dx.$$

7. $$d\log x = \frac{dx}{x}.$$

8. $$d\,e^x = e^x\,dx.$$

9. $$d\,a^x = a^x \log a\,dx.$$

10. $$d\sin^{-1} x = \frac{dx}{\sqrt{1-x^2}}.$$

11. $$d\cos^{-1} x = -\frac{dx}{\sqrt{1-x^2}}.$$

12. $$d\tan^{-1} x = \frac{dx}{1+x^2}.$$

13. $$d\cot^{-1} x = -\frac{dx}{1+x^2}.$$

It is important that the student gain facility in the use of the new results.

Example 1. Differentiate the function

$$u = \cos^{-1}\frac{x}{a}, \qquad a > 0.$$

Let
$$y = \frac{x}{a}.$$

Then
$$u = \cos^{-1} y,$$
$$du = d\cos^{-1} y$$
$$= -\frac{dy}{\sqrt{1-y^2}};$$
$$dy = \frac{dx}{a}.$$

Hence
$$-\frac{dy}{\sqrt{1-y^2}} = -\frac{\frac{dx}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = -\frac{dx}{\sqrt{a^2-x^2}},$$

and, finally,
$$d\cos^{-1}\frac{x}{a} = -\frac{dx}{\sqrt{a^2-x^2}}.$$

In abbreviated form,
$$d\cos^{-1}\frac{x}{a} = -\frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = -\frac{dx}{\sqrt{a^2-x^2}}.$$

Example 2. Differentiate the function

$$u = \tan^{-1}\frac{2x+1}{3}.$$

Here,
$$du = \frac{d\,\frac{2x+1}{3}}{1+\left(\frac{2x+1}{3}\right)^2} = \frac{\frac{2}{3}dx}{\frac{10+4x+4x^2}{9}} = \frac{3\,dx}{5+2x+2x^2},$$

or
$$\frac{du}{dx} = \frac{3}{5+2x+2x^2}.$$

The student should notice that the method used in the text for deriving the fundamental formulas of differentiation is not to be repeated in the applications. It is these formulas themselves that should be used. Thus, to solve Ex. 1 by writing

$$\cos u = \frac{x}{a}$$

and then differentiating would be logically irreproachable, but bad technique.

EXERCISES

Differentiate each of the following functions.

1. $u = \sin^{-1}\frac{x}{a}$. $\qquad \frac{du}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$, if $a > 0$.

2. $u = \tan^{-1}\frac{x}{a}$. $\qquad \frac{du}{dx} = \frac{1}{a^2 + x^2}$.

3. $u = \cot^{-1}\frac{x}{a}$. $\qquad du = -\frac{dx}{a^2 + x^2}$.

4. $u = \sin^{-1}(n \sin x)$. $\qquad \frac{du}{dx} = \frac{n \cos x}{\sqrt{1 - n^2 \sin^2 x}}$.

5. $u = \cos^{-1}\frac{1 - x}{2}$. $\qquad \frac{du}{dx} = \frac{1}{\sqrt{3 + 2x - x^2}}$.

6. $u = \sin^{-1}\frac{2x - 1}{\sqrt{2}}$.

7. $u = \cot^{-1}\frac{x + a}{b}$.

8. $u = \cot^{-1}\frac{1}{x}$.

9. $u = \tan^{-1}\frac{a}{x}$.

10. $u = \tan^{-1}\frac{2x}{1 - x^2}$. $\qquad \frac{du}{dx} = \frac{2}{1 + x^2}$.

11. $u = \tan^{-1}\left(x\frac{3 - x^2}{1 - 3x^2}\right)$. $\qquad \frac{du}{dx} = \frac{3}{1 + x^2}$.

12. $u = \sin^{-1}\frac{x - a}{x}$.

13. $u = \cos^{-1}\frac{x}{x + a}$.

14. $u = \sin^{-1}(2x\sqrt{1 - x^2})$. $\qquad \frac{du}{dx} = \frac{2}{\sqrt{1 - x^2}}, \quad |x| < \frac{1}{\sqrt{2}}$.

15. $t = \cos^{-1}\frac{s}{2}$. $\qquad \frac{ds}{dt} = -2\sin 2t.$

16. $t = \sin^{-1}\frac{s}{3}$. $\qquad \frac{ds}{dt} = 3\cos 3t.$

17. $t = \cos^{-1}\frac{s}{n} + \gamma$. $\qquad \frac{ds}{dt} = -n\sin n(t-\gamma)$.

18. $u = x\sin^{-1}x$.

19. $u = \frac{\tan^{-1}x}{x}$.

20. $u = \frac{1}{\sin^{-1}x}$. $\qquad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}\,(\sin^{-1}x)^2}$.

21. $u = a\cos^{-1}\frac{x-a}{a}$.

22. $u = \tan^{-1}\frac{x-a}{x+a}$.

23. $u = \cot^{-1}\frac{x+ab}{bx-a}$.

24 $u = \sin^{-1}\frac{ax+b}{bx+a}$.

25. $u = \sqrt{x^2-a^2} - a\cos^{-1}\frac{a}{x}$. $\qquad \frac{du}{dx} = \frac{\sqrt{x^2-a^2}}{x},\ a > 0.$

26. $u = \sin^{-1}\frac{x}{a} + \frac{\sqrt{a^2-x^2}}{x}$. $\qquad \frac{du}{dx} = \frac{\sqrt{a^2-x^2}}{x^2},\ a > 0.$

27. $u = \tan^{-1}\left(2\tan\frac{x}{2}\right)$. $\qquad \frac{du}{dx} = \frac{2}{5-3\cos x}$.

28. $u = \tan^{-1}(3\tan\theta)$. $\qquad \frac{du}{d\theta} = \frac{3}{5-4\cos 2\theta}$.

4. Continuation. Numerical Computation. By means of the Tables the numerical value of any of the functions of this chapter can be determined when a specific numerical value has been chosen for the independent variable. It is, however, an important aid to ease and security in such computations to be able, in advance, to make sure of the early significant figures and the location of the decimal point. There are two important geometrical methods for achieving this end. One is the representation of the trigonometric functions by suitable lines connected with the unit circle; the other consists in the graphs introduced above, in § 2.

First of all, however, it should be pointed out that there are two distinct problems. One is to find *all* values of x which satisfy such equations as

(*a*) $$\sin x = .2318;$$

(*b*) $$\cos x = -.4322;$$

(*c*) $$\tan x = -1.4861.$$

The other is to find the *principal value* of an inverse trigonometric function; for example,

$$\sin^{-1} .2318; \qquad \cos^{-1}(-.4322); \qquad \tan^{-1}(-1.4861)$$

The methods of treating these problems are identical.

First Geometric Method. Equations (*a*), (*b*), (*c*) can be solved graphically by the aid of the unit circle representation with an error corresponding to a degree or two, the results being expressed in radians if the problem comes from the Calculus.

For example, consider equation (*b*). The student should provide himself with an accurately drawn circle of his own construction, executed on the accurate centimeter-millimeter paper commercially procurable; the radius of the circle being 10 cm. and its center at a principal intersection of the rulings.

To solve equation (*b*), he will lay a straight-edge on his plate, parallel to the secondary (or y-) axis and at a distance of 4 cm., $3\frac{1}{4}$ mm. to the left of that axis. Marking the two points of intersection of the straight-edge with the circle by fine pencil lines easily erased, he now measures one of the acute angles involved by means of his protractor and thus determines the two solutions of (*b*) lying between 0° and 360° correct to minutes or thereabouts. By aid of the Tables the values can at once be converted into radian measure.

Arithmetic Solutions. From the figure before him the student now sees clearly a right triangle, one leg of which is known. The determination of the angle he needs is merely a problem in the solution of a right triangle by the tables, and

he proceeds to carry this work through to the degree of accuracy which the tables permit.

Equations (a) and (c) are treated in a similar manner. The point of this method is that the student is trained to *visualize a figure*, and not to try to remember a table that looks like

$$\sin A \qquad + \qquad + \qquad - \qquad -.$$

For, such tables vanish in a short time, and when the student needs his trigonometry in later work, he is helpless.

In terms of the inverse functions, this first problem consists in finding all the values of the multiple-valued function $\cos^{-1} x$ for the value of the variable, $x = -.4322$.

Second Geometric Method. This method consists in reading off from the graph the two values which lie between 0 and 2π, and then adding to these arbitrary positive or negative multiples of 2π.

The graph suggests, moreover, how to determine these values arithmetically by the aid of a table of sines or cosines of angles of the first quadrant. It also suggests a further refinement of the graphical method, of which the student will do well to avail himself, — namely, this. Let him make an accurate graph of the function

$$y = \sin x$$

on cm.-mm.-paper, taking 10 cm. as the unit and measuring the angle in radians, x ranging from 0 to $\pi/2$. This half-arch supplements the four graphs of the functions $\sin x$, $\cos x$, $\sin^{-1} x$, $\cos^{-1} x$ and serves as a 3-place table for determining their values (with a possible error of two or three units in the third place).

To sum up, then, there are two geometric methods; 1) the unit-circle method; 2) the graphs of the functions, the latter being supplemented by the 10-cm. graph just described. Either of the geometric methods suggests how to use the tables correctly and affords an altogether satisfactory check on the tables.

When the accurately drawn graphs are not at hand, freehand drawings indicate clearly how to use the tables with security and accuracy.

EXERCISES

1. Determine both in degrees and radians all values of x which satisfy the above equations (a), (b), (c), using each time all of the geometric methods set forth, and also the tables.

2. Find the value of each of the following functions. It is understood that the *principal value* is meant. Use first the method of the graphs. Then determine from the tables. Check by unit-circle and protractor.

i) $\sin^{-1}(-.1643)$; ii) $\cos^{-1}(.6417)$;

iii) $\tan^{-1}(-2.8162)$.

3. By means of a free-hand drawing of the graph estimate the value of each of the following functions. Remember that a curve recedes from its tangent very slowly near a point of inflection.

a) $\sin^{-1} .113$; b) $\tan^{-1}(-.214)$; c) $\cos^{-1} .172$;

d) $\tan^{-1}(-7.4)$; e) $\cot^{-1}(-.152)$; f) $\cos^{-1}(-.998)$;

g) $\sin^{-1}(-.21)$; h) $\sin^{-1} .89$; i) $\tan^{-1} 5.2$;

j) $\cot^{-1} 7.3$; k) $\cos^{-1}(-.138)$; l) $\sin^{-1}(-.138)$.

In what cases is your error large; in what, small?

5. Applications. The inverse trigonometric functions afford a convenient means of solving the following problem in Optics.

A ray of light is refracted in a prism. Show that its deviation from its original direction is least when the incident ray and the refracted ray make equal angles with the faces of the prism.

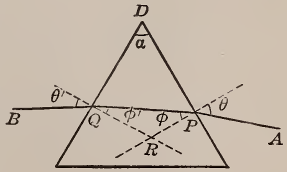

FIG. 73

The study of this problem has a vivid interest for the student who has seen the laboratory experiment of admitting a ray of sunlight into a darkened room, allowing it to pass through

a prism, thus being refracted, and throwing it finally, dispersed, on a screen.

Let AP be the incident ray; PQ, its path through the prism; and QB the ray which emerges. Then the deflection of PQ is obviously $\theta - \phi$ and the further deflection of QB is $\theta' - \phi'$; so that the total deflection, u, is:

(1) $$u = \theta - \phi + \theta' - \phi' = \theta + \theta' - (\phi + \phi').$$

On the other hand, the sum of the angles of the triangle PDQ is

$$\pi = \left(\frac{\pi}{2} - \phi\right) + \left(\frac{\pi}{2} - \phi'\right) + \alpha.$$

Hence

(2) $$\phi + \phi' = \alpha.$$

We can, therefore, write (1) in the form:

(3) $$u = \theta + \theta' - \alpha.$$

This is the quantity it is desired to make a minimum. θ and θ' are, however, connected by a relation which can be obtained as follows. We have by the law of refraction (cf. Chap. V, § 7):

(4) $$\frac{\sin\theta}{\sin\phi} = n, \qquad \frac{\sin\theta'}{\sin\phi'} = n.$$

Let $\nu = 1/n$. Then

(5) $$\sin\phi = \nu\sin\theta \qquad \text{or} \qquad \phi = \sin^{-1}(\nu\sin\theta).$$

Similarly,

(6) $$\sin\phi' = \nu\sin\theta' \qquad \text{or} \qquad \phi' = \sin^{-1}(\nu\sin\theta').$$

Substituting these values of ϕ and ϕ' in equation (2) we have the desired relation:

(7) $$\sin^{-1}(\nu\sin\theta) + \sin^{-1}(\nu\sin\theta') = \alpha.$$

Our problem now is completely formulated; it is: To make the function u given by (3) a minimum, when θ and θ' are connected by (7):

(8) $$\begin{cases} u = \theta + \theta' - \alpha, \\ \sin^{-1}(\nu\sin\theta) + \sin^{-1}(\nu\sin\theta') = \alpha. \end{cases}$$

Take θ as the independent variable. Then

$$\frac{du}{d\theta}=1+\frac{d\theta'}{d\theta}, \tag{9}$$

and the condition

$$\frac{du}{d\theta}=0 \qquad \text{gives} \qquad \frac{d\theta'}{d\theta}=-1.$$

Next, take the differential of each side of the second equation (8):

$$\frac{d(\nu \sin\theta)}{\sqrt{1-\nu^2\sin^2\theta}}+\frac{d(\nu\sin\theta')}{\sqrt{1-\nu^2\sin^2\theta'}}=0,$$

or

$$\frac{\nu\cos\theta\, d\theta}{\sqrt{1-\nu^2\sin^2\theta}}+\frac{\nu\cos\theta'\, d\theta'}{\sqrt{1-\nu^2\sin^2\theta'}}=0.$$

Hence

$$\frac{\cos\theta}{\sqrt{1-\nu^2\sin^2\theta}}+\frac{\cos\theta'}{\sqrt{1-\nu^2\sin^2\theta'}}\left(\frac{d\theta'}{d\theta}\right)=0. \tag{10}$$

But $d\theta'/d\theta=-1$. Consequently

$$\frac{\cos\theta}{\sqrt{1-\nu^2\sin^2\theta}}=\frac{\cos\theta'}{\sqrt{1-\nu^2\sin^2\theta'}}. \tag{11}$$

One solution of this equation is $\theta=\theta'$, — the solution demanded by the theorem. But conceivably there might be other solutions, and then it would not be clear which one of them makes u a minimum. We can readily show, however, that equation (11) has no further solutions. Square each side:

$$\frac{\cos^2\theta}{1-\nu^2\sin^2\theta}=\frac{\cos^2\theta'}{1-\nu^2\sin^2\theta'}.$$

Clear of fractions and express each cosine in terms of the sine:

$$(1-\nu^2\sin^2\theta')(1-\sin^2\theta)=(1-\nu^2\sin^2\theta)(1-\sin^2\theta').$$

Multiply out and suppress equal terms on the two sides:

$$-\sin^2\theta-\nu^2\sin^2\theta'=-\sin^2\theta'-\nu^2\sin^2\theta,$$

$$(\nu^2-1)\sin^2\theta=(\nu^2-1)\sin^2\theta'.$$

Hence

$$\sin^2\theta = \sin^2\theta', \qquad \sin\theta = \sin\theta',$$

and consequently the only angles of the first quadrant which can satisfy (11) are equal angles, $\theta = \theta'$.

From (5) and (6) it follows that $\phi = \phi'$. Hence, from (2)

$$\phi = \frac{\alpha}{2}, \qquad \text{and so} \qquad \theta = \sin^{-1}\left(n \sin\frac{\alpha}{2}\right),$$

$$u = 2\sin^{-1}\left(n\sin\frac{\alpha}{2}\right) - \alpha.$$

That u is a minimum, is clearly indicated by the laboratory experiment. It can be proven analytically as follows. From (9)

$$\frac{d^2u}{d\theta^2} = -\frac{d^2\theta'}{d\theta^2}.$$

Differentiate (10) as it stands; then, after the differentiation, set $d\theta'/d\theta = -1$ and $\theta = \theta'$. It is seen at once that

$$\frac{d^2\theta'}{d\theta^2} < 0, \qquad \text{hence} \qquad \frac{d^2u}{d\theta^2} > 0.$$

and u has a minimum.

EXERCISE

The bottom of a mural painting 4 ft. high is 12 ft. above the eye of the observer. How far back from the wall should he stand, in order that the angle subtended by the painting be as large as possible?

Suggestion. Take the distance, x, of the observer from the wall as the independent variable, and express the angle of elevation of the bottom and the top of the painting in terms of x.

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